Exact resolution of the Baxter equation for reggeized gluon interactions

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Abstract

The interaction of reggeized gluons in multi-colour QCD is considered in the Baxter-Sklyanin representation, where the wave function is expressed as a product of the Baxter functions $Q(\lambda)$ and a pseudo-vacuum state. We find n solutions of the Baxter equation for the composite state of n gluons with poles of rank r in the upper λ semi-plane and of rank $n-1-r$ in the lower λ semi-plane $(0 \leq r \leq n-1)$. These solutions are related by $n-2$ linear equations with coefficients containing $\coth(\pi\lambda)$. The poles cancel in the wave function, bilinear combination of holomorphic and anti-holomorphic Baxter functions, guaranteeing its normalizability. The quantization of the intercepts of the corresponding Regge singularities appears as a result of the physical requirement that the holomorphic energies for all Baxter functions are the same. The quantization results in simple properties of the zeroes of the Baxter functions. For illustration we calculate the intercepts of the reggeon states constructed from three and four gluons. In both cases the ground state of the system has conformal spin $|m - \widetilde{m}| = 1$. We calculate the anomalous dimensions of the corresponding operators for arbitrary α_s/ω.

1 Introduction

The leading logarithmic asymptotics (LLA) of scattering amplitudes in the Regge limit of high energy $\sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ is obtained by calculating and summing all contributions $(g^2 \ln s)^n$, where g is the QCD coupling constant. In this approximation the BFKL Pomeron is a composite state of two reggeized gluons [1]. The BFKL equation for the Pomeron wave function is closely related to the DGLAP equation for the parton distributions [2]. Next-to-leading corrections to its kernel were calculated in QCD [3] and in supersymmetric gauge theories [4].

The asymptotic behaviour $\propto s^{j_0}$ of scattering amplitudes is governed by the j-plane singularities of the t-channel partial waves $f_j(t)$

$$A(s,t) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dj}{2\pi i}\, \xi_j\; s^j\; f_j(t) \sim \xi_{1+\omega_0}\, s^{1+\omega_0}\,. \tag{1}$$

Here the contour of integration in j is situated to the right of the leading singularity $j_0 = 1+\omega_0$ of $f_j(t)$ $(j_0 < \sigma)$ and the signature factor is $\xi_{1+\omega_0} \simeq i^{n-1}$ for the t-channel exchange of n-reggeized gluons. Its intercept ω_0 for the corresponding Feynman diagrams is proportional to the ground

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state energy E_0 of the Schrödinger-like equation [1, 5]:

$$Hf = Ef \ , \ \omega_0 = -\frac{g^2}{8\pi^2} N_c \, E_0 \, . \tag{2}$$

Here the wave function f depends on the two-dimensional impact parameters $\overrightarrow{\rho_k}$ - positions of the reggeized gluons. It is convenient to introduce the holomorphic ($\rho_k = x_k + iy_k$) and anti-holomorphic ($\rho_k^* = x_k - iy_k$) coordinates and their corresponding momenta $p_k = i\frac{\partial}{\partial \rho_k}$ and $p_k^* = i\frac{\partial}{\partial \rho_k^*}$.

In multicolour QCD $N_c \to \infty$ the colour structure of the BFKL equation in LLA is significantly simplified. As a result, each reggeized gluon interacts only with its two neighbours [6]:

$$H \ = \ \frac{1}{2} \sum_{k=1}^{n} H_{k,k+1} \, . \tag{3}$$

Note, that for three gluon composite state, describing the Odderon responsible for the high energy behaviour of the differences of total cross-sections for particle-particle and particle-anti-particle interactions, this simplification is valid for arbitrary N_c [5, 7].

The Hamiltonian H has the property of holomorphic separability [6]:

$$H = \frac{1}{2}(h + h^*), \ \ [h, h^*] = 0 \, , \tag{4}$$

where the holomorphic and anti-holomorphic Hamiltonians

$$h = \sum_{k=1}^{n} h_{k,k+1} \, , \, h^* = \sum_{k=1}^{n} h^*_{k,k+1} \tag{5}$$

are expressed in terms of the BFKL operator [9]

$$h_{k,k+1} = \log(p_k) + \log(p_{k+1}) + \frac{1}{p_k} \log(\rho_{k,k+1}) p_k + \frac{1}{p_{k+1}} \log(\rho_{k,k+1}) p_{k+1} + 2\,\gamma \, . \tag{6}$$

Here $\rho_{k,k+1} = \rho_k - \rho_{k+1}$ and $\gamma = -\psi(1)$ is the Euler-Mascheroni constant.

The wave function $f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, ..., \overrightarrow{\rho_n}; \overrightarrow{\rho_0})$ of the colourless composite state described by the operator $O_{m,\widetilde{m}}(\overrightarrow{\rho_0})$ belongs to the principal series of unitary representations of the Möbius group [8]. For these representations the conformal weights

$$m = 1/2 + i\nu + n/2, \ \ \widetilde{m} = 1/2 + i\nu - n/2 \tag{7}$$

are expressed in terms of the anomalous dimension $\gamma = 1/2 + i\nu$ of $O_{m,\widetilde{m}}(\overrightarrow{\rho_0})$ and its integer conformal spin n. Furthermore, the eigenvalues of two Casimir operators M^2 and M^2 of the Möbius group are equal to $m(m-1)$ and $\widetilde{m}(\widetilde{m}-1)$, respectively.

Owing to the holomorphic separability of H, the wave function has the property of holomorphic factorization [6]:

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, ..., \overrightarrow{\rho_n}; \overrightarrow{\rho_0}) = \sum_{r,l} c_{r,l} \, f^r_m(\rho_1, \rho_2, ..., \rho_n; \rho_0) \, f^l_{\widetilde{m}}(\rho^*_1, \rho^*_2, ..., \rho^*_n; \rho^*_0) \, , \tag{8}$$

where r and l enumerate degenerate solutions of the Schrödinger equation in the holomorphic and anti-holomorphic sub-spaces:

$$\epsilon_m \, f_m = h \, f_m \quad , \quad \epsilon_{\widetilde{m}} \, f_{\widetilde{m}} = h^* \, f_{\widetilde{m}} \quad , \quad E_{m,\widetilde{m}} = \epsilon_m + \epsilon_{\widetilde{m}} \, . \tag{9}$$

Similarly to the case of two-dimensional conformal field theories, the coefficients $c_{r,l}$ are obtained imposing single-valuedness to $f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, ..., \overrightarrow{\rho_n}; \overrightarrow{\rho_0})$ as a function of the two-dimensional variables $\overrightarrow{\rho}_i$.

There are two different normalization conditions for the wave function [9]:

$$\|f\|_1^2 = \int \prod_{r=1}^{n} d^2\rho_r \left| \prod_{r=1}^{n} \rho_{r,r+1}^{-1} \; f \right|^2 , \;\; \|f\|_2^2 = \int \prod_{r=1}^{n} d^2\rho_r \left| \prod_{r=1}^{n} p_r \; f \right|^2 \tag{10}$$

compatible with the hermiticity of H. This property is related with the fact [9], that h commutes with the differential operator

$$A = \rho_{12}\rho_{23}...\rho_{n1}\, p_1 p_2 ... p_n \, . \tag{11}$$

Furthermore[10], there is a family $\{q_r\}$ of mutually commuting differential operators being the integrals of motion:

$$[q_r, q_s] = 0 \quad , \quad [q_r, h] = 0. \tag{12}$$

They can be obtained by the u-expansion of the transfer matrix for the XXX model [10]

$$T(u) = tr \; [L_1(u) L_2(u) ... L_n(u)] = \sum_{r=0}^{n} u^{n-r} \, q_r, \tag{13}$$

where the L-operators are

$$L_k(u) = \begin{pmatrix} u + \rho_{k0}\, p_k & -p_k \\ \rho_{k0}^2\, p_k & u - \rho_{k0}\, p_k \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} + \begin{pmatrix} 1 \\ \rho_{k0} \end{pmatrix} \begin{pmatrix} -\rho_{k0} & 1 \end{pmatrix} p_k \, . \tag{14}$$

In particular q_n is equal to A and q_2 is proportional to M^2.

The transfer matrix is the trace of the monodromy matrix $t(u)$:

$$T(u) = tr\, [t(u)] \quad , \quad t(u) = L_1(u) L_2(u) ... L_n(u) \, . \tag{15}$$

It can be shown [10, 11], that $t(u)$ satisfies the Yang-Baxter equation:

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u) \, , \tag{16}$$

where $l(w)$ is the L-operator for the well-known Heisenberg spin chain:

$$l_{s_1' s_2'}^{s_1 s_2}(w) = w\, \delta_{s_1'}^{s_1}\, \delta_{s_2'}^{s_2} + i\, \delta_{s_2'}^{s_1}\, \delta_{s_1'}^{s_2} \, . \tag{17}$$

The integrals of motion and the hamiltonian have an additional symmetry under the transformation [15]

$$p_k \to \rho_{k,k+1} \to p_{k+1} \tag{18}$$

combined with the operator transposition. This duality symmetry allows to relate the wave function of a composite state with the Fourier transformed wave function of another physical state. In particular, it gives the possibility to construct a new Odderon solution having the intercept exactly equal to unity [16]. The duality symmetry can be interpreted as a symmetry among the states constructed from the reggeons with positive and negative signatures [16]. Indeed, the Regge trajectories for these states with gluon quantum numbers are degenerated in multi-colour QCD.

2 Baxter-Sklyanin representation

Thus, the problem of finding solutions of the Schrödinger equation for the reggeized gluon interaction reduces to the search of a representation for the monodromy matrix satisfying the Yang-Baxter bilinear relations [10]. For this purpose the algebraic Bethe Ansatz is appropriate [11]. It is convenient to work in the conjugated space [13], where the monodromy matrix is parametrized as follows,

$$\widetilde{t}(u) = \widetilde{L}_n(u) ... \widetilde{L}_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} . \tag{19}$$

Here $\widetilde{L}_k(u)$ is given by

$$\widetilde{L}_k(u) = \begin{pmatrix} u + p_k \rho_{k0} & -p_k \rho_{k0}^2 \\ p_k \rho_{k0}^2 & u - p_k \rho_{k0} \end{pmatrix} . \tag{20}$$

Now the equation for the pseudo-vacuum state

$$C(u)\, |0\rangle^t = 0 \tag{21}$$

has the following solution [13]

$$|0\rangle^t = \prod_{k=1}^{n} \frac{1}{\rho_{k0}^2} . \tag{22}$$

In the total impact parameter space $\overrightarrow{\rho}$ the pseudo-vacuum wave function,

$$\Psi^{(0)}(\overrightarrow{\rho_1},\, \overrightarrow{\rho_2},\, \ldots, \overrightarrow{\rho_n}) = \prod_{k=1}^{n} \frac{1}{|\rho_{k0}|^4} . \tag{23}$$

is an eigenfunction of the transfer matrix,

$$[A(u) + D(u)]\, |0\rangle^t = [(u - i\,)^n + (u + i\,)^n]\ |0\rangle^t . \tag{24}$$

The pseudo-vacuum state does not belong to the principal series of unitary representations because it has the conformal weight $m = n$.

A powerful approach to construct the physical states in the framework of the Bethe Ansatz is based on the use of the Baxter equation for the Baxter function $Q(\lambda)$ [17]. The Baxter equation for the n-reggeon composite states can be written as follows (see [13, 19, 20])

$$\Lambda^{(n)}(\lambda;\ \vec{\mu})\, Q\,(\lambda;\ m, \vec{\mu}) = (\lambda + i)^n\ Q\,(\lambda + i;\ m, \vec{\mu}) + (\lambda - i)^n\ Q\,(\lambda - i;\ m, \vec{\mu})\ , \tag{25}$$

where $\Lambda^{(n)}(\lambda)$ is the polynomial

$$\Lambda^{(n)}(\lambda;\ \vec{\mu}) = \sum_{k=0}^{n} (-i)^k\, \mu_k\, \lambda^{n-k}\ ,\ \mu_0 = 2,\ \mu_1 = 0,\ \mu_2 = m(m-1)\,. \tag{26}$$

Here we assume in accordance with ref. [19], that the quantities $\mu_k = i^k\, q_k$ proportional to the eigenvalues q_k of the integrals of motion are real numbers, which is compatible with the single-valuedness condition of the wave functions.

The eigenfunctions of the holomorphic Schrödinger equation can be expressed through the Baxter function $Q(\lambda)$ using the Sklyanin Ansatz [18]:

$$f(\rho_1, \rho_2, ..., \rho_n) = Q(\widehat{\lambda}_1;\ m, \vec{\mu})\, Q(\widehat{\lambda}_2;\ m, \vec{\mu}) ... Q(\widehat{\lambda}_{n-1};\ m, \vec{\mu}) |0\rangle^t\,. \tag{27}$$

where $\widehat{\lambda}_r$ are the operator zeroes of the matrix element $B(u)$ of the monodromy matrix:

$$B(u) = -P \prod_{r=1}^{n-1} (u - \widehat{\lambda}_r) \,, \;\; P = \sum_{k=1}^{n} p_k \,. \tag{28}$$

These expressions are well defined because the operators $\widehat{\lambda}_r$ and P commute with each with other [18]

$$\left[\widehat{\lambda}_r \,, \widehat{\lambda}_s\right] = \left[\widehat{\lambda}_r \,, P\right] = 0. \tag{29}$$

In ref. [13] it was assumed without any convincing arguments, that $Q(\lambda)$ is an entire function in the complex λ plane. One of the purposes of our previous paper [19] was to find the class of functions to which the Baxter functions belong. For this purpose we performed an unitary transformation of the wave function of the composite state of n reggeized gluons from the coordinate representation to the Baxter-Sklyanin representation in which the operators $\widehat{\lambda_r}$ are diagonal [19] (see also [20]). The kernel of this transformation was expressed through the eigenfunctions of the operators $B(u)$ and $B^*(u)$. For the cases of the Pomeron and the Odderon the unitary transformation was constructed in an explicit form[19]. As a consequence of the single-valuedness condition for the kernel of the transformation one obtains the quantization of the arguments of the Baxter functions $Q(\lambda)$ and $Q(\lambda^*)$ in the holomorphic and anti-holomorphic sub-spaces (see [19, 20]):

$$\lambda = \sigma + i\,\frac{N}{2} \quad , \quad \lambda^* = \sigma - i\,\frac{N}{2} \quad , \tag{30}$$

where σ and N are real and integer numbers, respectively.

In ref. [19] we proposed a general method of solving the Baxter equation for the n-reggeon composite state. To begin with, the simplest n-reggeon solution of this equation was searched in the form of a sum over poles of orders from 1 up to $n-1$ situated in the upper semi-plane

$$Q^{(n-1)}\,(\lambda;\, m, \vec{\mu}) = \sum_{r=0}^{\infty} \frac{P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)}{(\lambda - i\,r)^{n-1}} \,, \tag{31}$$

where the $P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)$ are polynomials in λ of degree $n-2$. Inserting this ansatz in the Baxter equation leads to recurrence relations between the polynomials $P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)$ which allows us to calculate them successively starting from $P^{(n-2)}_{0;m,\vec{\mu}}(\lambda)$[19].

One can normalize this solution imposing the constraint

$$\lim_{\lambda\to 0} P^{(n-2)}_{0;m,\vec{\mu}}(\lambda) = 1 \tag{32}$$

Then, the remaining coefficients of the polynomial $P^{(n-2)}_{0,m,\vec{\mu}}(\lambda)$ are calculated from the condition

$$\lim_{\lambda\to\infty} Q\,(\lambda;\, m, \vec{\mu}) \sim \lambda^{-n+m} \tag{33}$$

which is a necessary condition in order $Q\,(\lambda;\, m, \vec{\mu})$ to be a solution of the Baxter equation at $\lambda \to \infty$. It is enough to require

$$\lim_{\lambda\to\infty} \lambda^{n-2} \sum_{r=0}^{\infty} \frac{P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)}{(\lambda - i\,r)^{n-1}} = 0\,. \tag{34}$$

This condition gives $n-1$ linear equations allowing to calculate all coefficients of the polynomial $P^{(n-2)}_{0,m,\vec{\mu}}(\lambda)$.

The existence of the second independent solution

$$Q^{(0)}\left(\lambda;\, m, \vec{\mu}\right) = Q^{(n-1)}\left(-\lambda;\, m, \vec{\mu^s}\right) = \sum_{r=0}^{\infty} \frac{P^{(n-2)}_{r;m,\vec{\mu^s}}(-\lambda)}{(-\lambda - i\,r)^{n-1}}\,, \tag{35}$$

where $\vec{\mu^s}$ has the components $\mu^s_k = (-1)^k \mu_k$, is related with the invariance of the Baxter equation under the simultaneous transformations

$$\lambda \to -\lambda\,,\;\; \mu \to \mu^s\,.$$

One can verify [19] that

$$Q^*\left(-\lambda;\, m, \vec{\mu^s}\right) = Q\left(\lambda^*;\, \widetilde{m}, \vec{\mu^s}\right)\,.$$

It turns out [19], that there is a set of Baxter functions $Q^{(t)}$ $(t = 0, 1, ..., n-1)$ having poles both in the upper and lower half-λ planes.

$$Q^{(t)}\left(\lambda;\, m, \vec{\mu}\right) = \sum_{r=0}^{\infty}\left[\frac{P^{(t-1)}_{r;m,\vec{\mu}}(\lambda)}{(\lambda - i\,r)^t} + \frac{P^{(n-2-t)}_{r;m,\vec{\mu^s}}(-\lambda)}{(-\lambda - i\,r)^{n-1-t}}\right]\,,$$

where the polynomials $P^{(t-1)}_r$ and $P^{(n-2-t)}_r$ are fixed by the recurrence relations following from the Baxter equation and from the condition that the above Baxter functions decrease at infinity more rapidly than λ^{-n+2}. These solutions $Q^{(t)}$ are linear combinations of $Q^{(n-1)}\left(\lambda;\, m, \vec{\mu}\right)$ and $Q^{(n-1)}\left(-\lambda;\, m, \vec{\mu^s}\right)$ with coefficients depending on $\coth(\pi\lambda)$ [19].

Using all these functions in the holomorphic and anti-holomorphic spaces one can construct in the total two-dimensional space (σ, N) the normalizable Baxter function $Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right)$ without poles at $\sigma = 0$ [19]

$$Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right) = \sum_{t,l} C_{t,l}\, Q^{(t)}\left(\lambda;\, m, \vec{\mu}\right)\, Q^{(l)}\left(\lambda^*;\, \widetilde{m}, \vec{\mu}\right) \tag{36}$$

by adjusting for this purpose the coefficients $C_{r,l}$.

Another problem in the Baxter approach is the calculation of the energy $E_{m,\widetilde{m}}$, because the expression suggested in ref. [13] leads to an infinite value of energy for the meromorphic Baxter functions. This problem was solved in ref. [19] by the unitary transformation of the Hamiltonian from the momentum to the Baxter-Sklyanin representation [19].

In the region, where gluon momenta are strongly ordered with the values,

$$|p_n| << |p_{n-1}| << ... << |p_1| = 1$$

this unitary transformation of the wave function $\Psi_{m,\widetilde{m}}$ is significantly simplified [19]

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1}, \dots, \overrightarrow{p_n}) \sim \prod_{k=1}^{n-1}\left(\int_{-\infty}^{+\infty} d\sigma_k \sum_{N_k=-\infty}^{+\infty} p_{k-1}^{i\lambda_k^*}\, p_{k-1}^{*-i\lambda_k}\right)\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda_1}, ..., \overrightarrow{\lambda}_{n-1})\,. \tag{37}$$

On the other hand, in this region

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1}, \dots, \overrightarrow{p_n}) \sim c_n\; |p_n|^2 \ln\frac{1}{|p_n|^2}\,, \tag{38}$$

where c_n is a constant [19]. Therefore $\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda}_{n-1})$ has first order poles at $\lambda_{n-1} = i$ and $\lambda^*_{n-1} = i$ for $N_{n-1} = 0$ and pole singularities at $\lambda_k = \lambda^*_k = 0$ $(k = 1, 2, ..., n-2)$. The action of the Hamiltonian H on the function $\Psi_{m,\widetilde{m}}$ near these singularities after the unitary transformation to the λ-space is drastically simplified [19]. Moreover, using the Sklyanin factorized expression for the wave function $\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda}_{n-1})$ we express the energy in terms of the behaviour of $Q(\overrightarrow{\lambda_{n-1}})$ near the pole at $\lambda_{n-1} = i$, $\lambda^*_{n-1} = i$ and obtain [19]

$$E = i \lim_{\lambda,\lambda^* \to i} \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\lambda^*} \ln\left[(\lambda - i)^{n-1}(\lambda^* - i)^{n-1}\,|\lambda|^{2\,n}\; Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right)\right] , \tag{39}$$

Since the function $Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right)$ is a bilinear combination of the independent Baxter functions $Q^{(t)}(\lambda)$ and $Q^{(l)}(\lambda^*)$ $(t\,, l\; = \;1, 2, ..., n)$, the holomorphic (anti-holomorphic) energies for all solutions should be the same

$$\epsilon_m = i \lim_{\lambda\to i} \frac{\partial}{\partial\lambda} \ln\left[\lambda^n\, P^{(t-1)}_{1;m,\vec{\mu}}(\lambda)\right] . \tag{40}$$

This leads to the quantization of the integrals of motion q_k $(k = 3, 4, ..., n)$ [19].

Note, that this condition can be obtained also as one of the consequences of the absence the poles in $Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right)$ at $\sigma = 0$ for $|N| > 0$. Because in the corresponding bilinear combinations there appear products of poles $(\lambda - ir)^{-s}$ and $(-\lambda^* - ir')^{s'}$, to cancel all singularities one should expand the functions $Q^{(t)}$, $Q^{(l)}$ in Laurent series near these poles and keep the necessary terms in such expansions. The coefficients of the Laurent series satisfy recurrence relations similar to those for the residues of the poles. However, it turns out that they **are not** proportional to the residues of the poles. We provide in Appendix B the explicit derivation for the quarteton (four Reggeons). Even a special choice of the values for the integrals of motion q_r does not lead to such proportionality because the recurrence relations are different at $k = 2$ [19].

We discuss below the solution of the Baxter equation by the method suggested in ref. [19]). Note, that for $n = 3$ this method gives results in a full agreement with those obtained by other approaches [19].

3 Meromorphic solutions of the Baxter equation

Let us rewrite the Baxter equation in real form introducing the new variable $x \equiv -i\lambda$,

$$\Omega(x, \vec{\mu})\, Q(x, \vec{\mu}) = (x + 1)^n\, Q(x + 1, \vec{\mu}) + (x - 1)^n\, Q(x - 1, \vec{\mu}) , \tag{41}$$

where

$$\Omega(x, \vec{\mu}) = \sum_{k=0}^{n} (-1)^k\, \mu_k\, x^{n-k} \tag{42}$$

and

$$\mu_0 = 2\,,\;\; \mu_1 = 0\,,\;\; \mu_2 = m(m - 1)\,,$$

assuming that the eigenvalues of the integrals of motion μ_k $(k > 2)$ are real numbers.

Then, the solution of the Baxter equation possessing poles only at $x = l = 0, 1, 2, ...$ can be written as follows,

$$Q^{(n-1)}(x, \vec{\mu}) = \sum_{l=0}^{\infty} \left[\frac{a_l(\vec{\mu})}{(x - l)^{n-1}} + \frac{b_l(\vec{\mu})}{(x - l)^{n-2}} + ... + \frac{z_l(\vec{\mu})}{x - l}\right] , \tag{43}$$

where the residues $a_l(\vec{\mu})$, $b_l(\vec{\mu})$, ..., $z_l(\vec{\mu})$ satisfy recurrence relations which can be easily obtained from the Baxter equation and its derivatives in the limit $x \to l$. Using these relations we can express all these residues in terms of $a_0(\vec{\mu})$, $b_0(\vec{\mu})$, ..., $z_0(\vec{\mu})$. In addition, imposing the Baxter equation at $x \to \infty$,

$$\lim_{x\to\infty} x^s \, Q^{(n-1)}(x,\vec{\mu}) = 0 \quad , \quad s = 1,2,\ldots,n-2\,, \tag{44}$$

fixes the parameters $b_0(\vec{\mu})$, ..., $z_0(\vec{\mu})$ in terms of $a_0(\vec{\mu})$. We then require the normalization condition

$$a_0(\vec{\mu}) = 1 \;,$$

without losing generality. In this normalization the holomorphic energy is (see [19])

$$\epsilon_n = \frac{b_1}{a_1} + n = b_0 - \frac{\mu_{n-1}}{\mu_n}\,.$$

Thus, the solution $Q^{(n-1)}(x,\vec{\mu})$ is uniquely defined and can be explicitly constructed.

Let us further introduce a set of the auxiliary functions for $r = 1,2,...,n-1$

$$f_r(x,\vec{\mu}) = \sum_{l=0}^{\infty}\left[\frac{\tilde{a}_l(\vec{\mu})}{(x-l)^r} + \frac{\tilde{b}_l(\vec{\mu})}{(x-l)^{r-1}} + ... + \frac{\tilde{g}_l(\vec{\mu})}{x-l}\right]\,, \tag{45}$$

where the coefficients $\tilde{a}_l, ..., \tilde{g}_l$ satisfy the same recurrent relations as $a_l, ..., z_l$ but with other initial conditions

$$\tilde{a}_0 = 1,\, \tilde{b}_0 = ... = \tilde{g}_0 = 0\,. \tag{46}$$

Note, that all functions $f_r(x,\vec{\mu})$ are expressed in terms of a subset of pole residues $\tilde{a}_l, ..., \tilde{z}_l$ for $f_{n-1}(x,\vec{\mu})$.

Now we write the Baxter function $Q^{(n-1)}(x,\vec{\mu})$ as a linear combination of $f_r(x,\vec{\mu})$

$$Q^{(n-1)}(x,\vec{\mu}) = \sum_{r=1}^{n-1} C_r(\vec{\mu})\, f_r(x,\vec{\mu})\,. \tag{47}$$

In order to impose the asymptotic condition (44) on $Q^{(n-1)}(x,\vec{\mu})$, we use the binomial series for the terms in eq.(45),

$$\frac{1}{(x-l)^j} = \sum_{n=0}^{\infty}\frac{l^n}{x^{n+j}}\frac{(j+n-1)!}{n!\,(j-1)!} = \sum_{s=j}^{\infty}\frac{l^{s-j}}{x^s}\frac{(s-1)!}{(s-j)!(j-1)!}\,.$$

We find a set of $n-2$ linear equations on the coefficients $C_r(\vec{\mu})$:

$$\sum_{r=1}^{n-1} G_{s,r}(\vec{\mu})\, C_r(\vec{\mu}) = 0 \quad , \quad s = 1,2,\ldots,n-2 \qquad C_{n-1}(\vec{\mu}) = 1\;,$$

where

$$G_{s,r}(\vec{\mu}) = \sum_{l=0}^{\infty}\left[\tilde{a}_l(\vec{\mu})\,\frac{(s-1)!\,l^{s-r}}{(s-r)!\,(r-1)!} + \tilde{b}_l(\vec{\mu})\,\frac{(s-1)!\,l^{s-r+1}}{(s-r+1)!\,(r-2)!} + \ldots + \tilde{g}_l(\vec{\mu})\,l^{s-1}\right]\,. \tag{48}$$

Obviously, the symmetric solution

$$Q^{(0)}(x,\vec{\mu}) = Q^{(n-1)}(-x,\vec{\mu^s}) \quad , \quad \text{where} \quad \mu^s_k \equiv (-1)^k \mu_k\;, \tag{49}$$

can be constructed in a similar way. It has poles at $x=-l$ $(l=0,1,...)$.

But there are other 'minimal' solutions $Q^{(t)}(x,\vec{\mu})$ $(t=1,2,...,n-2)$ of the Baxter equation having t-order poles at positive integer x and $(n-1-t)$-order poles at negative integer x [19]

$$Q^{(t)}(x,\vec{\mu}) = \sum_{r=1}^{t} C_r^{(t)}(\vec{\mu})\, f_r(x,\vec{\mu}) + \beta^{(t)}(\vec{\mu}) \sum_{r=1}^{n-1-t} C_r^{(n-1-t)}(\vec{\mu^s})\, f_r(-x,\vec{\mu^s})\,, \tag{50}$$

where the meromorphic functions $f_r(x,\vec{\mu})$ were defined above. Such form of the solution is related by the invariance of the Baxter equation under the substitution $x\to -x\,,\ \vec{\mu}\to\vec{\mu^s}$.

The coefficients $C_r^{(t)}(\vec{\mu})$,$C_r^{(n-1-t)}(\vec{\mu^s})$ and $\beta^t(\vec{\mu})$ are obtained imposing the asymptotic validity of the Baxter equation,

$$\lim_{x\to\infty} x^s\, Q^{(t)}(x,\vec{\mu}) = 0 \quad , \quad s=1,2,\ldots,n-2\,.$$

This leads to a system of $n-2$ linear equations

$$\sum_{r=1}^{t} G_{s,r}(\vec{\mu})\, C_r^{(t)}(\vec{\mu}) + (-1)^s\beta^{(t)}(\vec{\mu}) \sum_{r=1}^{n-1-t} G_{s,r}(\vec{\mu^s})\, C_r^{(n-1-t)}(\vec{\mu^s}) = 0\,,$$
$$s=1,2,\ldots,n-2\,, \tag{51}$$

where the matrix elements $G_{s,r}(\vec{\mu})$ are defined by eq.(48) and we normalize $Q^{(t)}(x,\vec{\mu})$ by choosing

$$C_t^{(t)}(\vec{\mu}) = C_{n-1-t}^{(n-1-t)}(\vec{\mu}) = 1\,. \tag{52}$$

Moreover, one can obtain the following relations using the symmetry of the Baxter equation,

$$Q^{(r)}(x,\vec{\mu}) = \beta^{(r)}(\vec{\mu})\; Q^{(n-1-r)}(-x,\vec{\mu^s})\,,\ \beta^{(r)}(\vec{\mu})\; \beta^{(n-1-r)}(\vec{\mu^s}) = 1\,,\ \beta^{(0)}(\vec{\mu}) = 1\,. \tag{53}$$

It is important to notice that three subsequent solutions $Q^{(r)}$ for $r=1,2,...,n-2$ are linearly related by

$$\left[\delta^{(r)}(\vec{\mu}) + \pi\,\cot(\pi x)\right]\, Q^{(r)}(x,\vec{\mu}) = Q^{(r+1)}(x,\vec{\mu}) + \alpha^{(r)}(\vec{\mu})\; Q^{(r-1)}(x,\vec{\mu})\,. \tag{54}$$

Indeed, the left and right-hand sides satisfy the Baxter equation everywhere including $x\to\infty$ and have the same singularities. Due to the uniqueness of the 'minimal' solutions the quantity $\cot(\pi x)\, Q^{(r)}(x,\vec{\mu})$ can be expressed as a linear combination of $Q^{(r-1)}(x,\vec{\mu})$, $Q^{(r)}(x,\vec{\mu})$ and $Q^{(r+1)}(x,\vec{\mu})$. Furthermore, the coefficient in front of $Q^{(r+1)}(x,\vec{\mu})$ is chosen to be 1 taking into account our normalization of $Q^{(r)}(x,\vec{\mu})$. In this normalization we obtain

$$\alpha^{(r)}(\vec{\mu}) = -\frac{\beta^{(r)}(\vec{\mu})}{\beta^{(r-1)}(\vec{\mu})}\,,\ \beta^{(r)}(\vec{\mu}) = (-1)^r \prod_{t=1}^{r} \alpha^{(t)}(\vec{\mu})\,. \tag{55}$$

Furthermore, the following relations hold for the coefficients $\alpha^{(r)}(\vec{\mu})$ and $\delta^{(r)}(\vec{\mu})$

$$\delta^{(r)}(\vec{\mu}) = -\delta^{(n-1-r)}(\vec{\mu^s}) \quad , \quad \alpha^{(r)}(\vec{\mu})\; \alpha^{(n-1-r)}(\vec{\mu^s}) = 1\,. \tag{56}$$

According to the above relations the functions $Q^{(r)}(x,\vec{\mu})$ $(r=1,2,\ldots,n-2)$ can be expressed as linear combinations of $Q^{(n-1)}(x,\vec{\mu})$ and $Q^{(0)}(x,\vec{\mu})$ with the coefficients being periodic functions of x [19]:

$$D\left[\vec{\delta}(\vec{\mu}),\ \vec{\alpha}(\vec{\mu})\,, \pi\cot(\pi x)\right]\; Q^{(r)}(x,\vec{\mu}) = \tag{57}$$

$$= D_0^{(r)}\left[\vec{\delta}(\vec{\mu}),\, \vec{\alpha}(\vec{\mu}), \pi\cot(\pi x)\right] Q^{(0)}(x,\vec{\mu}) + D_{n-1}^{(r)}\left[\vec{\delta}(\vec{\mu}), \vec{\alpha}(\vec{\mu}), \pi\cot(\pi x)\right] Q^{(n-1)}(x,\vec{\mu})$$

compatible with the relation (53).

The factor $D\left[\vec{\delta},\, \vec{\alpha},\, \pi\cot(\pi x)\right]$ can be written as the determinant of the matrix Λ

$$D\left[\vec{\delta},\, \vec{\alpha},\, \pi\cot(\pi x)\right] = \left|\left|\Lambda\left[\vec{\delta},\, \vec{\alpha},\, \pi\cot(\pi x)\right]\right|\right|. \tag{58}$$

The matrix Λ takes the form,

$$\Lambda\left[\vec{\delta},\, \vec{\alpha},\, \pi\cot(\pi x)\right] = \pi\, \cot(\pi x)\; I + \Delta\left[\vec{\delta},\, \vec{\alpha}\right]\;,\; I = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix} \tag{59}$$

and

$$\Delta\left[\vec{\delta},\, \vec{\alpha}\right] = \begin{pmatrix} \delta^{(1)} & -1 & 0 & ... & 0 \\ -\alpha^{(2)} & \delta^{(2)} & -1 & ... & 0 \\ 0 & -\alpha^{(3)} & \delta^{(3)} & ... & 0 \\ .... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \delta^{(n-2)} \end{pmatrix}. \tag{60}$$

In an analogous way $D_0^{(r)}\left[\vec{\delta}, \vec{\alpha}, \pi\cot(\pi x)\right]$ and $D_{n-1}^{(r)}\left[\vec{\delta}, \vec{\alpha}, \pi\cot(\pi x)\right]$ can be expressed in terms of the determinants of the matrices $\Lambda_0^{(r)}$ and $\Lambda_{n-1}^{(r)}$ of (lower) rank $n-3$ obtained from Λ by removing the column r and the first or the last line, respectively:

$$D_0^{(r)}\left[\vec{\delta}, \vec{\alpha}, \pi\cot(\pi x)\right] = (-1)^r\, \alpha^{(1)}\; \left|\left|\Lambda_0^{(r)}\right|\right|\;,\; D_{n-1}^{(r)}\left[\vec{\delta}, \vec{\alpha}, \pi\cot(\pi x)\right] = (-1)^r\; \left|\left|\Lambda_{n-1}^{(r)}\right|\right|$$

and

$$\Lambda_0^{(r)} = \begin{pmatrix} -\alpha^{(2)} & \delta^{(2)} + \pi\cot(\pi x) & ... & ... & 0 \\ 0 & -\alpha^{(3)} & ... & ... & 0 \\ ... & ... & \delta^{(r+1)} + \pi\cot(\pi x) & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & \delta^{(n-2)} + \pi\cot(\pi x) \end{pmatrix},$$

$$\Lambda_{n-1}^{(r)} = \begin{pmatrix} \delta^{(1)} + \pi\cot(\pi x) & -1 & ... & ... & 0 \\ -\alpha^{(2)} & \delta^{(2)} + \pi\cot(\pi x) & ... & ... & 0 \\ ... & ... & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & -1 \end{pmatrix}.$$

Let us introduce for n reggeons the notation:

$$Q^{(r)}(x,\, \vec{\mu}) = a^{(r)}[\vec{\mu}, \pi\cot(\pi x)]\; Q^{(n)}[x,\, \vec{\mu}] + b^{(r)}[\vec{\mu}, \pi\cot(\pi x)]\, Q^{(0)}[x,\, \vec{\mu}]$$

and for $n+1$ reggeons:

$$Q^{(r)}(x,\, \vec{\mu}) = a'^{(r)}[\vec{\mu}, \pi\cot(\pi x)]\; Q^{(n+1)}(x,\, \vec{\mu}) + b'^{(r)}[\vec{\mu}, \pi\cot(\pi x)]\; Q^{(0)}(x,\, \vec{\mu})\;.$$

Then, the following relations are valid

$$a'^{(r)}[\vec{\mu}, \pi\cot(\pi x)] = a^{(r)}[\vec{\mu}, \pi\cot(\pi x)]\; a'^{(n)}[\vec{\mu}, \pi\cot(\pi x)]\;,$$

$$b'^{(r)}[\vec{\mu}, \pi\cot(\pi x)] = b^{(r)}[\vec{\mu}, \pi\cot(\pi x)] + a^{(r)}[\vec{\mu}, \pi\cot(\pi x)]\; b'^{(n)}[\vec{\mu}, \pi\cot(\pi x)]\;, \qquad (61)$$

where

$$a'^{(n)}[\vec{\mu}, \pi\cot(\pi x)] = \frac{1}{\delta^{(n)}(\vec{\mu}) + \pi\cot(\pi x) - \alpha^{(n)}(\vec{\mu})}\;,$$

$$b'^{(n)}[\vec{\mu}, \pi\cot(\pi x)] = b^{(n-1)}[\vec{\mu}, \pi\cot(\pi x)]\; \alpha^{(n)}(\vec{\mu})\, a'^{(n)}[\vec{\mu}, \pi\cot(\pi x)]\;. \qquad (62)$$

Notice that the linear relations (54) among $Q^{(r)}(x,\,\vec{\mu})$, $Q^{(r+1)}(x,\,\vec{\mu})$ and $Q^{(r-1)}(x,\,\vec{\mu})$ are similar to the recurrence relations for orthogonal polynomials $P_r(z)$ if we substitute $\frac{1}{\pi}\cot(\pi x)$ by a variable z. The Baxter functions also belong to an orthonormalizable set of functions.

We use below the formulae of this section for the numerical calculations of the intercepts of the composite states constructed from reggeized gluons.

4 Zeroes of the Baxter function and the quantization of the integrals of motion

The zeroes of the Baxter function are very important, because their position λ_k is fixed by the Bethe equations and their knowledge gives a possibility to write the wave function of the composite states in the framework of the Bethe Ansatz as an (infinite) product of the differential operators $B(\lambda_k)$ applied to the pseudo-vacuum state. The positions of the zeroes of $Q(\lambda)$ for the Pomeron wave function was investigated in our previous paper [19]. For the case of the composite states constructed from $n > 2$ reggeons the number of the 'minimal' solutions of the Baxter equation is n and their linear combinations have rather complicated sets of zeroes.

However, certain linear combinations of functions $Q^{(r)}(x,\,\vec{\mu})$ have zeroes which are situated at equidistant points $x_k = x_0 + k,\ k \in \mathcal{Z}$. Indeed, according to relations (54) among the Baxter functions $Q^{(r)}(x,\,\vec{\mu})$, $Q^{(r+1)}(x,\,\vec{\mu})$ and $Q^{(r-1)}(x,\,\vec{\mu})$ such situation is valid for solutions of the equation

$$Q^{(r+1)}(x,\vec{\mu}) + \alpha^{(r)}(\vec{\mu})\, Q^{(r-1)}(x,\vec{\mu}) = 0\,, \qquad (63)$$

where $r = 1, 2, \ldots, n-2$. Some of the solutions of eq.(63) coincide with the zeroes of the function $Q^{(r)}(x,\,\vec{\mu})$ while other roots are situated at the equidistant points

$$x_k^{(r)}(\vec{\mu}) = k - \frac{1}{\pi}\,\mathrm{arccot}\left[\frac{\delta^{(r)}(\vec{\mu})}{\pi}\right]\,, \quad k \in \mathcal{Z}. \qquad (64)$$

For the eigenstates of the Hamiltonian the parameters $\vec{\mu}$ are quantized in accordance with the physical requirement [19] that the holomorphic energies for different Baxter functions are the same. It is equivalent to the condition, that all parameters $\delta^{(r)}(\vec{\mu})$ vanish

$$\delta^{(r)}(\vec{\mu}) = 0\;, \qquad (65)$$

because otherwise, the energies for the solutions $Q^{(r)}(x,\vec{\mu})$ and $Q^{(r+1)}(x,\vec{\mu})$ would not coincide. Eq.(64) then implies that for quantized $\vec{\mu}$ the above linear combination of $Q^{(r+1)}(x,\vec{\mu})$ and $Q^{(r-1)}(x,\vec{\mu})$ has a sequence of zeroes at the points

$$x_k = k + \frac{1}{2}\,, \ \ k \in \mathcal{Z}.$$

Let us now consider linear combinations of the Baxter functions $Q^{(n-1)}(x,\vec{\mu})$ and $Q^{(0)}(x,\vec{\mu})$ with their poles situated only at positive and negative integer points, respectively. It is obvious from the previous section that there are $n-2$ different combinations of these functions

$$\Phi^{(t)}(x,\vec{\mu}) = Q^{(n-1)}(x,\vec{\mu}) + c^{(t)}(\vec{\mu})\, Q^{(0)}(x,\vec{\mu}) \tag{66}$$

having equidistant zeroes at the points $x = x_k$ where (see (58))

$$D\left[\vec{\delta}(\vec{\mu}),\, \vec{\alpha}(\vec{\mu})\,, \pi\cot(\pi x)\right] = 0\;. \tag{67}$$

The last equation has $n-2$ different solutions

$$z^{(t)}(\vec{\mu}) = \pi\cot\left[\pi x^{(t)}(\vec{\mu})\right]\;,\;\; t = 1, 2, ...n-2 \tag{68}$$

and for each solution there is a linear combination of the Baxter functions $Q^{(n-1)}(x,\vec{\mu})$ and $Q^{(0)}(x,\vec{\mu})$ with the relative coefficient

$$c^{(t)}(\vec{\mu}) = \frac{D_0^{(r)}\left[\vec{\delta}(\vec{\mu}), \vec{\alpha}(\vec{\mu})\;, z^{(t)}(\vec{\mu})\right]}{D_{n-1}^{(r)}\left[\vec{\delta}(\vec{\mu}), \vec{\alpha}(\vec{\mu}), z^{(t)}(\vec{\mu})\right]}\,. \tag{69}$$

Note, that $c^{(t)}(\vec{\mu})$ does not depend on the parameter r, if there is no accidental degeneracy.

In the case, when

$$\delta^{(r)}(\vec{\mu}) = 0 \tag{70}$$

for all $r = 1, 2, ..., n-2$, which corresponds to the quantization of the integrals of motion $\vec{\mu}$, the determinant $D\left[\vec{\delta}(\vec{\mu}),\, \vec{\alpha}(\vec{\mu}),\, \pi\cot(\pi x)\right]$ is an even (odd) function of its argument $\pi\cot(\pi x)$ for even (odd) n. Therefore, the equidistant zeroes of two different functions $\Phi^{(t)}(x,\vec{\mu})$ (or zeroes of the same function $\Phi^{(t)}(x,\vec{\mu})$) have opposite signs (modulo an integer number):

$$x^{(t)}(\vec{\mu}) = -x^{(n-2-t)}(\vec{\mu})\;. \tag{71}$$

For odd n one function has a sequence of zeroes at

$$x_k = k + \frac{1}{2}\,.$$

In a general case of n reggeized gluons we can calculate the coefficients $\alpha^{(r)}(\vec{\mu})$ in the recurrent relations and the quantized values of $\vec{\mu}$ only by knowing $Q^{(n)}$, $Q^{(0)}$ and all their linear combinations with constant coefficients which have the equidistant zeroes with above properties. Let us consider several examples:

For $n = 3$ we have only one function with equidistant zeroes $x_k = k + 1/2$

$$Q(x) = Q^{(2)}(x,\mu) + \alpha^{(1)}(\mu)\; Q((0)(x,\mu)\,.$$

For $n = 4$ there are two functions,

$$Q_\pm = Q^{(3)} \pm \alpha^{(1)}\sqrt{\alpha^{(2)}}\, Q^{(0)}$$

with zeroes at $\pi\cot\pi x = \pm\sqrt{\alpha_2}$, respectively. For $n = 5$ there are also two functions with equidistant zeroes:

$$Q_1 = Q^{(4)} + \alpha^{(1)}\,\alpha^{(2)}\, Q^{(0)}\;,\;\; Q_1 = Q^{(4)} - \alpha^{(1)}\,\alpha^{(3)}\, Q^{(0)}$$

with the equidistant zeroes at $\pi\cot\pi x = \pm\sqrt{\alpha^{(2)} + \alpha^{(3)}}$ and $\pi\cot\pi x = 0$, respectively. In these cases, one can calculate all $\alpha^{(r)}$ and the quantized values of $\vec{\mu}$ computing the corresponding combinations of $Q^{(n)}$ and $Q^{(0)}$ with the equidistant zeroes.

5 Anomalous dimensions of quasi-partonic operators

The Q^2-dependence of the inclusive probabilities $n_i(x, \ln Q^2)$ for finding a parton i with momentum fraction x inside a large hadron with large momentum $|\overrightarrow{p}| \to \infty$ can be found from the DGLAP evolution equation [2]. The eigenvalues of its integral kernels describing the inclusive parton transitions $i \to k$ coincide with the matrix elements $\gamma_j^{ki}(\alpha)$ of the anomalous dimension matrix for the twist-2 operators O^j with Lorentz spins $j = 2, 3, \ldots$. These operators are bilinear in the gluon ($i = g$) or quark ($i = q$) fields. For example, the twist-2 gluon operator with Lorentz spin j can be written as,

$$O^j_{.....} = n^{\mu_1} n^{\mu_2} ... n^{\mu_j} \, tr \; G_{\rho\mu_1} D_{\mu_2} D_{\mu_3} ... D_{\mu_{j-1}} G_{\rho\mu_j} \,, \tag{72}$$

where $D_\mu = \partial_\mu + g\, V_\mu$ is the covariant derivative, V_μ the gluon field and $G_{\rho\mu} = \frac{1}{g}\left[D_\rho \,, D_\mu\right]$ the corresponding field tensor.

The symmetric tensor $O_{\mu_1\mu_2...\mu_j}$ is multiplied by the light-cone vectors n_{μ_r}

$$n_\mu = q_\mu + x\, p_\mu \,, \; n_\mu^2 = 0 \,, \; p_\mu^2 \simeq 0 \,, \; q_\mu^2 = -Q^2, \; x = \frac{Q^2}{2pq} \,,$$

The n_{μ_r} are constructed in the case of the deep-inelastic ep scattering from the momenta p_μ and q_μ of the initial proton and virtual photon, respectively.

The matrix elements of the operators $O^j_{.....}$ between the hadron states are renormalized as functions of the growing ultraviolet cut-off Q^2. For example, in the case of the pure Yang-Mills theory with gauge group $SU(N_c)$ we have

$$\left\langle p \left| O^j_{.....} \right| p \right\rangle \sim \exp\left(\int_{Q_0^2}^{Q^2} \gamma_j(\alpha_s(Q'^2))\, d \ln Q'^2 \right) \,, \;\; \alpha_s(Q^2) \simeq \frac{4\pi}{\beta_2 \ln \frac{Q^2}{\Lambda_s^2}} \,, \;\; \beta_2 = \frac{11}{3} N_c - \frac{2}{3} n_f \,.$$

where $\Lambda_s \simeq 200\, Mev$ is the QCD parameter and the anomalous dimension $\gamma_j(\alpha)$ can be calculated perturbatively as

$$\gamma_j(\alpha) = \sum_{k=1}^{\infty} C_j^{(k)} \left(\frac{\alpha N_c}{\pi} \right)^k \,. \tag{73}$$

In particular, to the lowest order

$$C_j^{(1)} = \Psi(1) - \Psi(j-1) - \frac{2}{j} + \frac{1}{j+1} - \frac{1}{j+2} + \frac{11}{12} \,.$$

The gluon anomalous dimension is singular in the non-physical point $\omega = j - 1 \to 0$. In this limit one can calculate it to all orders of perturbation theory [8]

$$\gamma_\omega = \frac{\alpha N_c}{\pi\omega} - \Psi''(1) \left(\frac{\alpha N_c}{\pi\omega} \right)^4 + ... \tag{74}$$

from the eigenvalue of the BFKL equation in LLA [1] at $n = 0$:

$$\omega_{BFKL} = \frac{\alpha\, N_c}{\pi} \left[2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma) \right] \,. \tag{75}$$

Notice that the coefficients of $\left(\frac{\alpha N_c}{\pi\omega}\right)^2$ and $\left(\frac{\alpha N_c}{\pi\omega}\right)^3$ **exactly** vanish in eq.(74). Here $\Psi''(1) = -2\zeta(3)$.

Indeed, the Green function satisfying the inhomogeneous BFKL equation can be written as follows [8]

$$< \phi(\overrightarrow{\rho_1})\, \phi(\overrightarrow{\rho_2})\, \phi(\overrightarrow{\rho_{1'}})\, \phi(\overrightarrow{\rho_{2'}}) \ > = \sum_n \int_{-\infty}^{\infty} d\nu C(\nu,n) \int d^2\rho_0\, \frac{E_{\nu,n}(\overrightarrow{\rho_{10}},\overrightarrow{\rho_{20}})\, E^*_{\nu,n}(\overrightarrow{\rho_{1'0}},\overrightarrow{\rho_{2'0}})}{\omega-\omega^0(n,\nu)}\,, \quad (76)$$

where $\omega^0(n,\nu)$ is its eigenvalue and $C(\nu,n)$ is fixed by the completeness condition for the eigenfunctions

$$E_{\nu,n}(\overrightarrow{\rho_{10}},\overrightarrow{\rho_{20}}) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho^*_{12}}{\rho^*_{10}\rho^*_{20}}\right)^{\widetilde{m}}\,,\ m=\frac{1}{2}+i\nu+\frac{n}{2}\,,\ \widetilde{m}=\frac{1}{2}+i\nu-\frac{n}{2}\,. \quad (77)$$

In the limit $|\rho_{1'2'}| \to 0$ one can perform a Wilson expansion for the product of the fields $\phi(\overrightarrow{\rho_{1'}})$ and $\phi(\overrightarrow{\rho_{2'}})$. In this case the integral over $\overrightarrow{\rho_0}$ can be calculated by extracting the factor $E_{\nu,n}(\overrightarrow{\rho_{10}},\overrightarrow{\rho_{20}})$ from the integral at the point $\rho_0=\rho_{1'}$ and by using the relation

$$\int d^2\rho_0\, E^*_{\nu,n}(\overrightarrow{\rho_{1'0}},\overrightarrow{\rho_{2'0}}) \sim \rho^m_{1'2'}\,\rho^{*\widetilde{m}}_{1'2'} \sim \left(\frac{\rho_{1'2'}}{\rho^*_{1'2'}}\right)^{\frac{n}{2}} |\rho_{1'2'}|^{2\Gamma}\,,\ \ \Gamma=\frac{m+\widetilde{m}}{2}=1-\gamma\,.$$

Thanks to this simplification at $|\rho_{1'2'}| \to 0$ one can then shift the integration contour in ν into the lower half of the ν-plane up to the first pole of $[\omega-\omega^0(n,\nu)]^{-1}$ with negative imaginary part of [8, 21]:

$$\lim_{|\rho_{1'2'}|\to 0} < \phi(\overrightarrow{\rho_1})\, \phi(\overrightarrow{\rho_2})\, \phi(\overrightarrow{\rho_{1'}})\, \phi(\overrightarrow{\rho_{2'}}) \ > \sim$$

$$\sum_n e^{i\varphi n} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\widetilde{C}(\gamma,n)\ d\,\gamma}{\omega-\omega^0(n,\nu)} \left|\frac{\rho_{12}\,\rho_{1'2'}}{\rho_{11'}\rho_{22'}}\right|^{2(1-\gamma)}\,,\ \ \gamma=\frac{1}{2}-i\nu\,, \quad (78)$$

where $\varepsilon \to 0^+$ and

$$e^{i\varphi} = \sqrt{\frac{\rho^*_{11'}\rho^*_{22'}\rho_{12}\rho_{1'2'}}{\rho^*_{12}\rho^*_{1'2'}\rho_{11'}\rho_{22'}}}\,.$$

Note, that in accordance with the fact, that the Green function includes the external gluon propagators, it behaves in the momentum space as $|k|^{-4+2\gamma}$ at large gluon virtualities $|k|^2$.

In particular, for $n=0$ and $\gamma \to 0$ one can obtain the above expansion (74) in powers of $\frac{\alpha}{\pi\omega}$ for the position γ_ω of the pole $(\gamma-\gamma_\omega)^{-1}$. It is important, that for real ω the pole is situated on the real axis. Therefore, the description of the corresponding state in the framework of the Möbius group approach requires the exceptional series of unitary representations (contrary to the Regge kinematics, where the principal series is used). For the exceptional series the anomalous dimension

$$\gamma = 1-\frac{m+\widetilde{m}}{2}$$

is real.

One can calculate from the BFKL equation also anomalous dimensions of higher twist operators by solving the eigenvalue equation near other singular points $\gamma=-k$ $(k=1,2,...)$ or by including in it a dependence from the conformal spin $|n|$, which also leads to a shift of the pole position to $\gamma=-|n|/2$ (see [4]).

But a more important problem is the calculation of the anomalous dimensions for the so-called quasi-partonic operators (QPO) [22] constructed from several gluonic fields. Indeed, the contribution of these operators at $j \to 1$ is responsible for the unitarization of structure functions at high energies. The simplest operator of such type is the product of the twist-2 gluon operators

$$O^j = \prod_{r=1}^{p} O^{j_r}_{....}\,, \quad j = \sum_{r=1}^{p} j_r = p + \omega\,, \; \omega = \sum_{r=1}^{p} \omega_r\,. \tag{79}$$

In the limit $N_c \to \infty$ this operator is multiplicatively renormalized [24] and its dimension is the sum of dimensions of its factors (including their anomalous dimensions $\gamma(\omega_r)$)

$$\Gamma = p - \gamma\,, \; \gamma = \sum_{r=1}^{p} \gamma(\omega_r).\,, \tag{80}$$

where in the expression for the total dimension Γ we neglected the small contribution $\frac{\omega}{2}$.

To investigate the multi-Pomeron configuration, let us write the hamiltonian describing the corresponding composite state in LLA as a sum of independent BFKL hamiltonians

$$H = \sum_{r=1}^{p} H_{a_r b_r}$$

and its eigenfunction as a product of Pomeron wave functions:

$$\Psi(\rho_{a_1}, \rho_{b_1}, \rho_{0_1}; ...; \rho_{a_p}, \rho_{b_p}, \rho_{0_p}) = \prod_{r=1}^{p} \left(\frac{\rho_{a_r 0_r}\rho_{b_r 0_r}}{\rho_{a_r b_r}}\right)^{m_r} \left(\frac{\rho^*_{a_r 0_r}\rho^*_{b_r 0_r}}{\rho^*_{a_r b_r}}\right)^{\widetilde{m}_r}\,,$$

where ρ_{a_r}, ρ_{b_r} and ρ_{0_r} are the coordinates of gluons a_r, b_r and Pomerons 0_r, respectively, and the quantities m_r and $\widetilde{m}_r$ are their conformal weights. The wave function Ψ belongs to a reducible representation of the Möbious group and can be expanded in a sum of irreducible representations with the use of the Clebsch-Gordon coefficients $C^{m,\widetilde{m}}_{m_1\widetilde{m}_1;...;m_r\widetilde{m}_r}$ [23]

$$\prod_{r=1}^{p} O_{m_r\widetilde{m}_r}(\overrightarrow{\rho}_{0_r}) = \sum_{m,\widetilde{m}} \int d^2\rho_0\, C^{m,\widetilde{m}}_{m_1\widetilde{m}_1;...;m_r\widetilde{m}_r}(\overrightarrow{\rho}_{0_1},\, \overrightarrow{\rho}_{0_2}, ...,\, \overrightarrow{\rho}_{0_p};\, \overrightarrow{\rho}_0)\; \Phi^{m,\widetilde{m}}(\overrightarrow{\rho_0}).$$

The contribution to the scattering amplitude in the coordinate space from each irreducible component can be written as follows

$$\prod_{r=1}^{p}\left[\sum_{n_r=0}^{\infty}\int_{-\infty}^{\infty} d\nu_r \int d^2\rho_{0_r} \left(\frac{\rho_{a_r b_r}}{\rho_{a_r 0_r}\rho_{b_r 0_r}}\right)^{m_r} \left(\frac{\rho^*_{a_r b_r}}{\rho^*_{b_r 0_r}\rho^*_{a_r 0_r}}\right)^{\widetilde{m}_r}\right] \frac{C^{m,\widetilde{m}}_{m_1\widetilde{m}_1;...;m_r\widetilde{m}_r}(\overrightarrow{\rho}_{0_1},\, \overrightarrow{\rho}_{0_2}, ...,\, \overrightarrow{\rho}_{0_p};\, \overrightarrow{\rho}_0)}{\omega - \sum_{s=1}^{p}\omega(n_s,\nu_s)}\,.$$

In the Regge regime m_r and $\widetilde{m}_r$ belong to the principal series of the unitary representations of the Möbious group and therefore m and $\widetilde{m}$ also belong to the same series [23]. It means, that the position ω_0 of the t-channel partial wave singularity related to the asymptotics of the cross-section $\sigma_t \sim s^{\omega_0}$ equals $\omega_0 = p\,\omega_{BFKL}$.

In the deep-inelastic regime the essential intervals are small $|\rho_{00_r}| \sim 1/Q \ll |\rho_{a_r b_t}|$. From dimensional considerations we obtain at large $|Q|$ after integration over the essential region of ρ_{0_r} a power asymptotics of the scattering amplitude

$$A^{(m,\widetilde{m})} \sim \prod_{r=1}^{p}\left[\sum_{n_r=0}^{\infty}\int_{\infty}^{\infty} d\nu_r \left(\frac{\rho_{a_r b_r}}{\rho_{a_r 0}\rho_{b_r 0}}\right)^{m_r} \left(\frac{\rho^*_{a_r b_r}}{\rho^*_{b_r 0}\rho^*_{a_r 0}}\right)^{\widetilde{m}_r}\right] \frac{C^{m,\widetilde{m}}_{m_1\widetilde{m}_1;...;m_p\widetilde{m}_p}\, |Q|^{-2(p-\gamma)}\, e^{in\varphi}}{\omega - \sum_{s=1}^{p}\omega(n_s,\nu_s)}, \tag{81}$$

where $e^{i\varphi}=\sqrt{Q/Q^*}$, $n=\frac{1}{2}\sum_{r=1}^{p}(m_r-\widetilde{m}_r)$ and

$$\gamma=p-\frac{1}{2}\sum_{r=1}^{p}(m_r+\widetilde{m}_r) \tag{82}$$

is the anomalous dimension of a composite operator which turns out to be real and small for small g^2/ω. This asymptotics is in agreement with the known result [23], that in the product of the exceptional representations with real $\gamma_r=1-\frac{m_r+\widetilde{m}_r}{2}>0$ there is a continuous spectrum of unitary representations of the principal series and only one representation from the exceptional series having

$$\gamma=\sum_{r=1}^{p}\gamma_r\,. \tag{83}$$

Since $\omega(n,\nu)$ has a negative second derivative $\omega''_{\gamma\gamma}(n,\nu)$ at $0<\gamma<1$, we obtain after the integration over ν_r with the use of the saddle-point method the following result

$$A^{(m,\widetilde{m})}\sim\prod_{r=1}^{p}\left[\left(\frac{\rho_{a_rb_r}}{\rho_{a_r0}\rho_{b_r0}}\right)^{\frac{m}{p}}\,\left(\frac{\rho^*_{a_rb_r}}{\rho^*_{b_r0}\rho^*_{a_r0}}\right)^{\frac{\widetilde{m}}{p}}\right]|Q|^{-2(p-\gamma)}\,, \tag{84}$$

where

$$\gamma=p\,\omega^{(-1)}(\frac{\omega}{p})\,. \tag{85}$$

and $\omega^{(-1)}(\omega)$ is the inverse function to $\omega=\omega_{BFKL}(\gamma)$.

In particular, for very large Q^2 corresponding to $\gamma\to 0$ and $n=0$ we have [24]

$$\gamma=\frac{\alpha_s N_c}{\pi\omega}\,p^2\,. \tag{86}$$

It is possible to calculate also a correction of the relative order N_c^{-2} to this expression [24].

Let us consider now for the multi-colour QCD the high energy asymptotics of the irreducible Feynman diagrams in which each of the n reggeized gluons interacts with two neighbours. In the Born approximation the corresponding Green function is the product of free gluon Green functions $\prod_{r=1}^{n}\ln|\rho_r-\rho'_r|^2$. In LLA $\frac{\alpha_s}{\omega}\sim 1$ it can be written.as follows (cf. [8])

$$<\ \phi(\overrightarrow{\rho_1})...\phi(\overrightarrow{\rho_n})\,\phi(\overrightarrow{\rho'_1})...\phi(\overrightarrow{\rho'_n})\ >=$$

$$\sum_n\int_{-\infty}^{\infty}d\nu\sum_{\mu_3}\sum_{\mu_4}...\sum_{\mu_n}C_{m,\widetilde{m};\mu_3,...}\int d^2\rho_0\,\frac{\Psi_{m,\widetilde{m};\mu_3,...}(\overrightarrow{\rho}_1,...,\,\overrightarrow{\rho}_n;\,\overrightarrow{\rho}_0)\,\Psi^*_{m,\widetilde{m};\mu_3,...}(\overrightarrow{\rho}'_1,...,\,\overrightarrow{\rho}'_n;\,\overrightarrow{\rho}'_0)}{\omega-\omega(m,\widetilde{m};\mu_3,...,\mu_n)}\,, \tag{87}$$

where $\omega(m,\widetilde{m};\mu_3,\mu_2,...,\mu_n)\sim\alpha_s$ are the eigenvalues of H depending on the quantized integrals of motion $\mu_3,\mu_2,...,\mu_n$ and $\Psi_{m,\widetilde{m};\mu_3,...,\mu_n}(\overrightarrow{\rho}_1,\ \overrightarrow{\rho}_2,...,\ \overrightarrow{\rho}_n;\ \overrightarrow{\rho}_0)$ are the corresponding eigenfunctions. From the Möbius invariance of the Schrödinger equation we obtain

$$\Psi_{m,\widetilde{m};\mu_3,\mu_2,...,\mu_n}(\overrightarrow{\rho}_1,...,\,\overrightarrow{\rho}_n;\,\overrightarrow{\rho}_0)\sim\left(\frac{\rho_{12}\rho_{23}\cdots\rho_{n1}}{\rho^2_{10}\rho^2_{20}\cdots\rho^2_{n0}}\right)^{\frac{m}{n}}\left(\frac{\rho^*_{12}\rho^*_{23}\cdots\rho^*_{n1}}{\rho^{*2}_{10}\rho^{*2}_{20}\cdots\rho^{*2}_{n0}}\right)^{\frac{\widetilde{m}}{n}}f_{m,\overrightarrow{m}}(\overrightarrow{x}_1,...,\,\overrightarrow{x}_{n-2}) \tag{88}$$

where $\overrightarrow{x}_r$ (x_r and x^*_r) are independent anharmonic ratios of ρ_k and ρ^*_k for $k=0,1,...,n$.

In the Bjorken region, where

$$|\rho'_r - \rho'_s| \sim Q^{-1} \ll |\rho_r - \rho_s| \ ,$$

the essential domain is $|\rho_0 - \rho'_r| \sim 1/Q$ and therefore from dimensional considerations

$$< \ \phi(\overrightarrow{\rho'_1})...\phi(\overrightarrow{\rho'_n}) \ >\sim \sum_n \int_{-\infty}^{\infty} d\nu \sum_{\mu_3} \sum_{\mu_4} ... \sum_{\mu_n} C_{m,\widetilde{m};\mu_3,...} \ \frac{\Psi_{m,\widetilde{m};\mu_3,...}(\overrightarrow{\rho}_1, ...; \ \overrightarrow{\rho}_0) \, Q^m Q^{\widetilde{m}}}{\omega - \omega(m,\widetilde{m};\mu_3,...,\mu_n)} \, . \qquad (89)$$

It means, that in this limit the contour of integration over ν should be shifted to the lower half of the complex plane till the first pole of $[\omega - \omega(m,\widetilde{m};\mu_3,\mu_2,...,\mu_n)]^{-1}$.

For small coupling constants α_s this singularity for n reggeized gluons is situated near the pole of $\omega(m,\widetilde{m};\mu_3,\mu_2,...,\mu_n)$. The position of the leading pole is

$$\frac{m+\widetilde{m}}{2} = \frac{n}{2} - \gamma^{(n)} \quad , \quad \gamma^{(n)} = c^{(n)} \, \frac{\alpha_s \, N_c}{\omega} + O\left(\left[\frac{\alpha_s \, N_c}{\omega}\right]^2\right) . \qquad (90)$$

This relation is in agreement with the above estimate of the anomalous dimension for the diagrams with the t-channel exchange of p numbers of the BFKL ladders because here $n = 2p$ and $c^{(2p)} = p^2$. Moreover, it can be obtained from the solution of the equation for matrix elements of quasi-partonic operators written with a double-logarithmic accuracy in ref. [25]. Let us derive a similar equation starting from the Schrödinger equation for the composite states of reggeized gluons in the multi-colour QCD.

In the case of two reggeized gluons in the momentum space we obtain for $|p'_1| \simeq |p'_2| \gg |p_1| \, , \, |p_2|$ the equation

$$\omega \Psi(\overrightarrow{p}_1, \overrightarrow{p}_2) = -\frac{\alpha_s N_c \, (\overrightarrow{p}_1, \overrightarrow{p}_2)}{\pi \, |p_1|^2 \, |p_2|^2} \int_{\max(|p_1|^2, \, |p_2|^2)}^{\infty} d \, |p'_1|^2 \int_0^{2\pi} \frac{d\varphi_{1'}}{2\pi} \, \Psi(\overrightarrow{p}'_1, -\overrightarrow{p}'_1) \, .$$

Note, that the Bethe-Salpeter amplitude $\Psi(\overrightarrow{p}_1, \overrightarrow{p}_2)$ contains the gluon propagators $|p_1|^{-2} \, , |p_2|^{-2}$. The solution of this equation is

$$\Psi(\overrightarrow{p}_1, \overrightarrow{p}_2) \sim \frac{(\overrightarrow{p}_1, \overrightarrow{p}_2)}{|p_1|^2 \, |p_2|^2} \left(\frac{|p_1|}{|Q|}\right)^{-\gamma} \, , \ \gamma = \frac{\alpha_s N_c}{\pi \, \omega} \, .$$

In the case of n reggeized gluons in the multi-colour QCD apart from the product of the propagators $\prod_{r=1}^n |p_r|^{-2}$ one can extract from the amplitude due to its gauge invariance also the momenta $p_r^{\mu_r}$ for each gluon

$$\Psi(\overrightarrow{p}_1, \overrightarrow{p}_2, ..., \overrightarrow{p}_n) \sim \prod_{r=1}^n \frac{p_r^{\mu_r}}{|p_r|^2} \ f_{\mu_1,\mu_2,...,\mu_n} \ . \qquad (91)$$

The factor $\prod_{r=1}^n p_r^{\mu_r} / \, |p_r|^2$ leads after the Fourie transformation to a singularity of $\frac{m+\widetilde{m}}{2}$ near $n/2$ and $f_{\mu_1,\mu_2,...,\mu_n}$ is the tensor in a two-dimensional transverse subspace depending on $\xi_r = \ln \, p_i^2$. The Schrödinger equation for this tensor takes the form

$$\omega \, f_{\mu_1,\mu_2,...,\mu_n} = \frac{\alpha_s N_c}{4\pi} \sum_{r=1}^n \delta_{\mu_r \mu_{r+1}} \delta_{\mu'_r \mu'_{r+1}} \int_{\xi_r}^{\infty} d\xi'_r \int_{\xi_{r+1}}^{\infty} d\xi'_{r+1} \, \delta(\xi'_r - \xi'_{r+1}) \ f_{\mu_1,\mu_2,...\mu'_r,\mu'_{r+1}...,\mu_n} \, . \quad (92)$$

In principle this equation gives a possibility to calculate the anomalous dimension $\gamma = \gamma(\omega) = c_n\,\alpha_s/\omega$. A similar equation is discussed in ref.[25]. In particular for the Odderon ($n = 3$) it turns out, that $c_3 = 0$ according to an unpublished result of M. Ryskin and A. Shuvaev. We confirm this result below [see eq.(109)] by solving the Baxter equation and finding a pole singularity near of $\frac{m+\widetilde{m}}{2} = 2$ (instead of 3/2 as it could be expected from the above relation). For $n = 4$ in accordance with the general formula $\frac{m+\widetilde{m}}{2} = \frac{n}{2} - \gamma^{(n)}$ we find the pole singularity near $\frac{m+\widetilde{m}}{2} = 2$ as shown in eq.(122). Moreover, similar to the case of the BFKL Pomeron the anomalous dimensions γ_3 and γ_4 are calculated for arbitrary α/ω, which is important for finding multi-reggeon contributions to the deep-inelastic processes at small Bjorken's variable x. We plot in fig.1 and 3 the dependence of ω/α on $m = \widetilde{m}$ for the odderon and in fig. 5 for the four reggeon state.

6 The solutions of the Baxter equation for the odderon

We construct the explicit solutions of the Baxter equation for the odderon ($n = 3$) following the general method presented in sec. 3.

The Baxter equation for the odderon takes the real form (41)

$$B_3\,(x;\,m,\mu) \equiv \left[2x^3 + m(m-1)x + \mu\right] Q\,(x;\,m,\mu)$$

$$-(x+1)^3\,Q\,(x+1;\,m,\mu) - (x-1)^3\,Q\,(x-1;\,m,\mu) = 0\,. \tag{93}$$

where

$$q_3 = i\mu\,,\;\; Im\,(\mu) = 0\,. \tag{94}$$

μ must be real in order to obtain single-valued odderon wave functions in coordinate space.

The auxiliary functions f_r (45) for the odderon take the form

$$f_2\,(x;\,m,\,\mu) = \sum_{l=0}^{\infty}\left[\frac{a_l(m,\,\mu)}{(x-l)^2} + \frac{b_l(m,\,\mu)}{x-l}\right]\,,$$

$$f_1\,(x;\,m,\,\mu) = \sum_{l=0}^{\infty}\frac{a_l(m,\mu)}{x-l}\,. \tag{95}$$

The residues satisfy the recurrence relations

$$(r+1)^3\,a_{r+1}(m,\,\mu) = \left[2\,r^3 + m(m-1)\,r - \mu\right]\,a_r(m,\,\mu) - (r-1)^3\,a_{r-1}(m,\,\mu)\,,$$

$$(r+1)^3\,b_{r+1}(m,\,\mu) = \left[2\,r^3 + m(m-1)\,r - \mu\right]\,b_r(m,\,\mu) - (r-1)^3\,b_{r-1}(m,\,\mu)$$

$$+\left[6\,r^2 + m(m-1)\right]\,a_r(m,\,\mu) - 3\,(r+1)^2\,a_{r+1}(m,\,\mu) - 3\,(r-1)^2\,a_{r-1}(m,\,\mu)\,. \tag{96}$$

We can choose,

$$a_0(m,\,\mu) = 1 \quad , \quad b_0 = 0\,, \tag{97}$$

and **all** the coefficients $a_r(m,\,\mu)$ and $b_r(m,\,\mu)$ become uniquely determined by eqs.(96). In particular,

$$a_1(m,\,\mu) = -\mu \quad , \quad 8\,a_2(m,\,\mu) = -\mu\,[2 + m(m-1) - \mu]\,,$$

$$b_1(m,\,\mu) = m(m-1) + 3\mu\;, \tag{98}$$

$$8\,b_2(m,\,\mu) = [2 + m(m-1) - \mu]\;b_1(m,\,\mu) - \mu\,[6 + m(m-1)] - 12\,a_2(m,\,\mu)\;.$$

Notice that the coefficients of the leading pole singularities obey identical recurrence equations in the context of these Baxter equations.

Thanks to eq.(96), $B_3\,(x;\,m,\mu)$ is an entire function of x. We find from eq.(93) that it has generically the form

$$B_3\,(x;\,m,\mu) = [6 - m(m-1)]\lim_{x\to\infty}[x\,Q\,(x;\,m,\,\mu)]\;.$$

The Baxter equation is fulfilled provided the limit in the r. h. s. vanishes.

Notice that a solution of the Baxter equation multiplied by a periodic function of x with period 1 is again a solution of the Baxter equation. For example, the function $\pi\cot\pi x$ which is a constant at infinity. Furthermore, if $Q\,(x;\,m,\mu)$ is a solution of eq.(93), then $Q\,(-x;\,m,-\mu)$ is also a solution of eq.(93).

We can now form linear combinations of the auxiliary functions $f_1\,(x;\,m,\,\mu)$ and $f_2\,(x;\,m,\,\mu)$ in order to obtain solutions of the Baxter equation. We consider[19]:

$$\begin{aligned} Q^{(2)}\,(x;\,m,\mu) &= f_2\,(x;\,m,\,\mu) + B(m,\mu)\;f_1\,(x;\,m,\,\mu) \\ Q^{(1)}\,(x;\,m,\mu) &= f_1\,(x;\,m,\,\mu) + C(m,\mu)\,Q_1\,(-x;\,m,\,-\mu) \end{aligned} \tag{99}$$

As noticed above, the Baxter equation $B_3\,(x;\,m,\mu) = 0$ is fulfilled at infinity provided the coefficients of x^{-1} in $Q^{(2)}\,(x;\,m,\mu)$ and in $Q^{(1)}\,(x;\,m,\mu)$ vanish for large x,

$$\begin{aligned} &\sum_{r=0}^{\infty} b_r(m,\,\mu) + B(m,\mu)\,\sum_{r=0}^{\infty} a_r(m,\mu) = 0\;, \\ &\sum_{r=0}^{\infty} a_r(m,\mu) - C(m,\mu)\,\sum_{r=0}^{\infty} a_r(m,-\mu) = 0\;. \end{aligned} \tag{100}$$

This gives for the coefficients,

$$B(m,\mu) = -\frac{\sum_{r=0}^{\infty} b_r(m,\,\mu)}{\sum_{r=0}^{\infty} a_r(m,\mu)} \quad , \quad C(m,\mu) = \frac{\sum_{r=0}^{\infty} a_r(m,\mu)}{\sum_{r=0}^{\infty} a_r(m,-\mu)}\;. \tag{101}$$

Therefore, the solutions $Q^{(2)}\,(x;\,m,\mu)$ and $Q^{(1)}\,(x;\,m,\mu)$ are completely determined.

The solutions $Q^{(2)}\,(x;\,m,\mu)\,,\;Q^{(0)}\,(x;\,m,\mu) = Q^{(2)}\,(-x;\,m,-\mu)$ and $Q^{(1)}\,(x;\,m,\mu)$ are related by the linear equation[19]

$$[\pi\cot\pi x + \delta(m,\mu)]\,Q^{(1)}\,(x;\,m,\mu) = Q^{(2)}\,(x;\,m,\mu) - C(m,\mu)\;Q^{(2)}\,(-x;\,m,-\mu) \tag{102}$$

This is a special case of eq.(54) for $n = 3,\; r = 1$.

The coefficient $\delta(m,\mu)$ can be written as,

$$\delta(m,\mu) = -B(m,-\mu) - \frac{1}{\mu\,C(m,-\mu)}\sum_{r=0}^{\infty}\frac{a_r(m,\mu)}{r+1} - \sum_{r=1}^{\infty}\frac{a_r(m,-\mu)}{r}$$

The energy is given for the odderon by[19]

$$E = \lim_{x,x^*\to 1} \left\{ \frac{\partial}{\partial x} \ln\left[(x-1)^2\, x^3\, Q^{(2)}(x;\, m,\, \mu)\right] + \frac{\partial}{\partial x^*} \ln\left[(x^*-1)^2\, x^{*3}\, Q^{(2)}(x^*;\, \widetilde{m},\, -\mu)\right] \right\} . \tag{103}$$

Thus, the energy is expressed in terms of the behavior of the Baxter function $Q^{(2)}(x, m, \mu)$ near $x = 1$.

We obtain for the energy of the solutions $Q^{(2)}\,(x;\, m, \mu)$ and $Q^{(1)}\,(x;\, m, \mu)\ \pi\, \cot \pi x$ from eqs.(99) and (103),

$$E^{(2)}(m, \widetilde{m}, \mu) = B(m, \mu) + B(\widetilde{m}, -\mu) \ , \tag{104}$$

$$E^{(1)}(m, \widetilde{m}, \mu) = 3 + \frac{1}{\mu} \left[\sum_{r=2}^{\infty} \frac{a_r(m, \mu)}{r-1} + C(m, \mu) \sum_{r=0}^{\infty} \frac{a_r(m, -\mu)}{r+1} \right] + (\mu \to -\mu) \ .$$

Since the energy must be the same for all independent Baxter solutions, we get as eigenvalue condition,

$$B(m, \mu) + B(\widetilde{m}, -\mu) = -\sum_{r\neq 0}^{\infty} \frac{a_r(m, \mu)}{r} + \frac{C(m, \mu)}{\mu} \sum_{r=0}^{\infty} \frac{a_r(m, \mu)}{r+1} + (\mu \to -\mu) \ . \tag{105}$$

This equation fixes the possible values of μ for given $m, \widetilde{m}$. Notice that eq.(105) is identical to require $\delta(m, \mu) + \delta(m, -\mu) = 0$.

We found for $m = \widetilde{m} = 1/2$: the first roots numerically from the above equations as,

$$\mu_1 = 0.205257506\ldots \quad , \quad \mu_2 = 2.3439211\ldots \quad , \quad \mu_3 = 8.32635\ldots \quad , \quad \mu_4 = 20.080497\ldots \quad , \ldots$$

with the corresponding energies

$$E_1 = 0.49434\ldots \quad , \quad E_2 = 5.16930\ldots \quad , \quad E_3 = 7.70234\ldots \quad , \quad E_4 = 9.46283\ldots \quad , \ldots \tag{106}$$

We have followed the eigenvalue $E_1,\ \mu_1$ as a function of m for $0 < m < \frac{1}{2}$. The result is plotted in fig. 1. Notice that only $m = 0,\ 1$ and $\frac{1}{2}$ are physical values.

It should be noticed that the energy **vanishes** at $m = 0$. This could be inferred from the fact that $E(m = 0, \mu \equiv 0) = 0$ in the expression for $E(m, \mu \equiv 0)$ given by eq.(74) of ref.[19].

$$E(m, \mu \equiv 0) = \frac{\pi}{\sin(\pi m)} + \psi(m) + \psi(1-m) - 2\psi(1)\,.$$

It should be noticed that $E(m, \mu \equiv 0)$ **does not** describe an eigenvalue **except** at $m = \mu = 0$.

For the eigenvalue 1 we obtain numerically,

$$E_1(m) \overset{m\to 0}{=} 2.152\ldots\ m - 2.754\ldots\ m^2 + \mathcal{O}(m^3) \quad , \quad \mu_1(m) \overset{m\to 0}{=} 0.375\ldots\sqrt{m} - 0.0228\, m + \mathcal{O}(m^{\frac{3}{2}}) \tag{107}$$

Notice that all quantities are here functions of m through the combination $m(1-m)$. Therefore, they are invariant under the exchange $m \Leftrightarrow 1-m$. Therefore, eqs.(107) yields the behaviour of E_1 and μ_1 near $m = 1$.

$$E_1(m) \quad \overset{m\to 1}{=} \quad 2.152\ldots\ (1-m) - 2.754\ldots\ (1-m)^2 + \mathcal{O}\left[(m-1)^3\right] \ ,$$

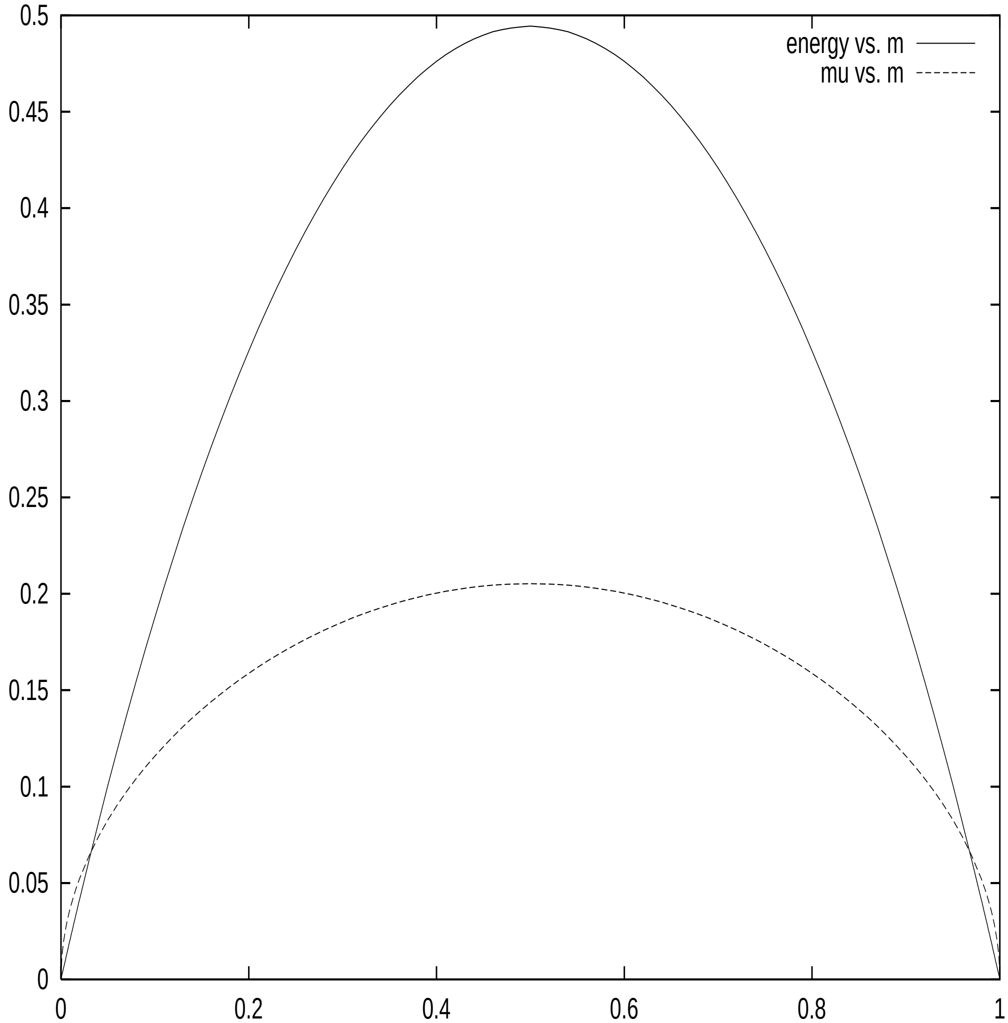

Figure 1: The energy and μ as functions of m for the odderon eigenvalue $E_1,\ \mu_1$ in the interval $0 < m < 1$. The picture is symmetric under $m \Leftrightarrow 1 - m$.

$$\mu_1(m) \stackrel{m\to 1}{=} 0.375\ldots\sqrt{1-m} - 0.0228\,(1-m) + \mathcal{O}\left[(1-m)^{\frac{3}{2}}\right] . \tag{108}$$

The state with $m = 1$ and $\widetilde{m} = 0$ (or viceversa) is therefore the **ground state** of the odderon. It is a zero energy eigenstate obviously below the eigenstates (106).

Furthermore, we can consider the states with conformal spin $n = 1$ where

$$m = 1 + i\,\nu \quad , \quad \widetilde{m} = i\,\nu$$

For small ν we can compute the energy of such state from eqs.(107) and (108) with the result

$$E = E_1(m) + E_1(\widetilde{m}) = 5.51\ldots\nu^2 + \mathcal{O}\left(\nu^4\right) .$$

The zero energy state that we find for $\nu = 0$ is the one found in ref. [16] by a different approach.

We have also followed the eigenstate 1 as a function of ν for real ν and

$$m = \frac{1}{2} + i\,\nu$$

The result is plotted in fig. 2.

We find for small ν

$$E_1(\nu) = 0.49434\ldots + 1.8179\ldots\ \nu^2 + \mathcal{O}(\nu^4) \quad , \quad \mu_1(\nu) = 0.205257\ldots + 0.48579\ldots\ \nu^2 + \mathcal{O}(\nu^4)$$

We continue the first eigenstate for $m > 1$ turning μ to purely imaginary. We plot in figs. 3 and 4 the energy and Imμ as functions of m for the eigenvalue 1 in the interval $1 \le m \le 2$.

Near $m = 2$ the energy eigenvalue diverges while μ tends to zero. We find the following behaviours near $m = 2$,

$$E_1(m) \stackrel{m\to 2}{=} \frac{2}{m-2} + 1 + 2 - m + \mathcal{O}\left[(m-2)^2\right] \quad , \quad i\mu \stackrel{m\to 2}{=} 2 - m - \frac{3}{2}\,(m-2)^2 + \mathcal{O}\left[(m-2)^3\right] \tag{109}$$

7 The solutions of the Baxter equation for four reggeons: the quarteton

The Baxter equation for the quartet (four reggeons state) takes the form

$$\begin{aligned} B_4\,(x;\,m,\mu,q_4) &\equiv \left[2x^4 + m(m-1)x^2 - \mu x + q_4\right] Q\,(x;\,m,\mu,q_4) \\ &-(x+1)^4\, Q\,(x+1;\,m,\mu,q_4) - (x-1)^4\, Q\,(x-1;\,m,\mu,q_4) = 0\,. \end{aligned} \tag{110}$$

where $q_3 = i\mu\,,\ Im\,(\mu) = 0$. A new integral of motion q_4 appears here. μ and q_4 must be real in order to obtain single-valued wave functions in coordinate space.

Notice that if $Q\,(x;\,m,\mu,q_4)$ is a solution of eq.(110) then $Q\,(-x;\,m,-\mu,q_4)$ is also a solution of eq.(110).

Following the general method presented in sec. III and [19] we seek the solutions of the Baxter equation for the quartet as a series of poles. We start by finding the recurrence relations

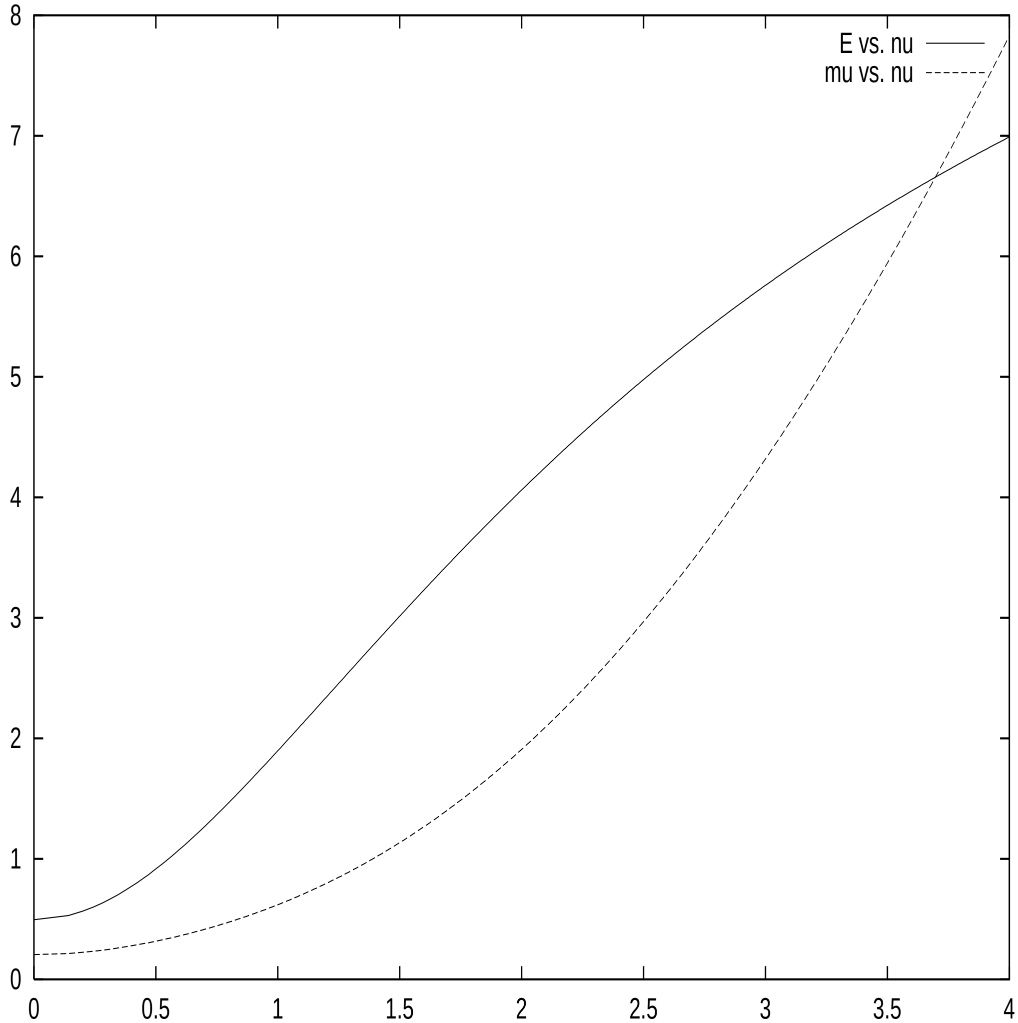

Figure 2: The energy and μ as functions of real ν for the odderon eigenvalue E_1, μ_1 setting $m = \frac{1}{2} + i\,\nu$.

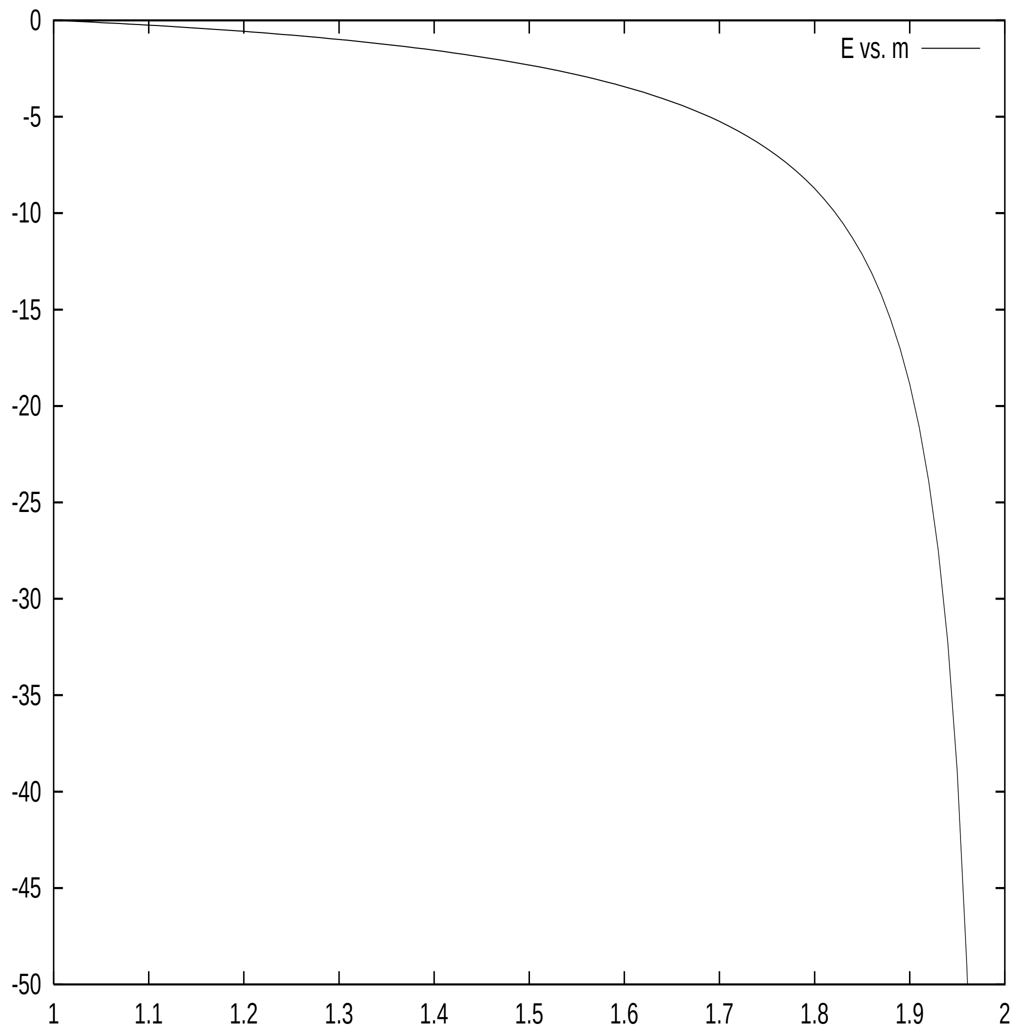

Figure 3: The odderon energy as a function of m for the eigenvalue 1 in the interval $1 \leq m \leq 2$ and imaginary μ.

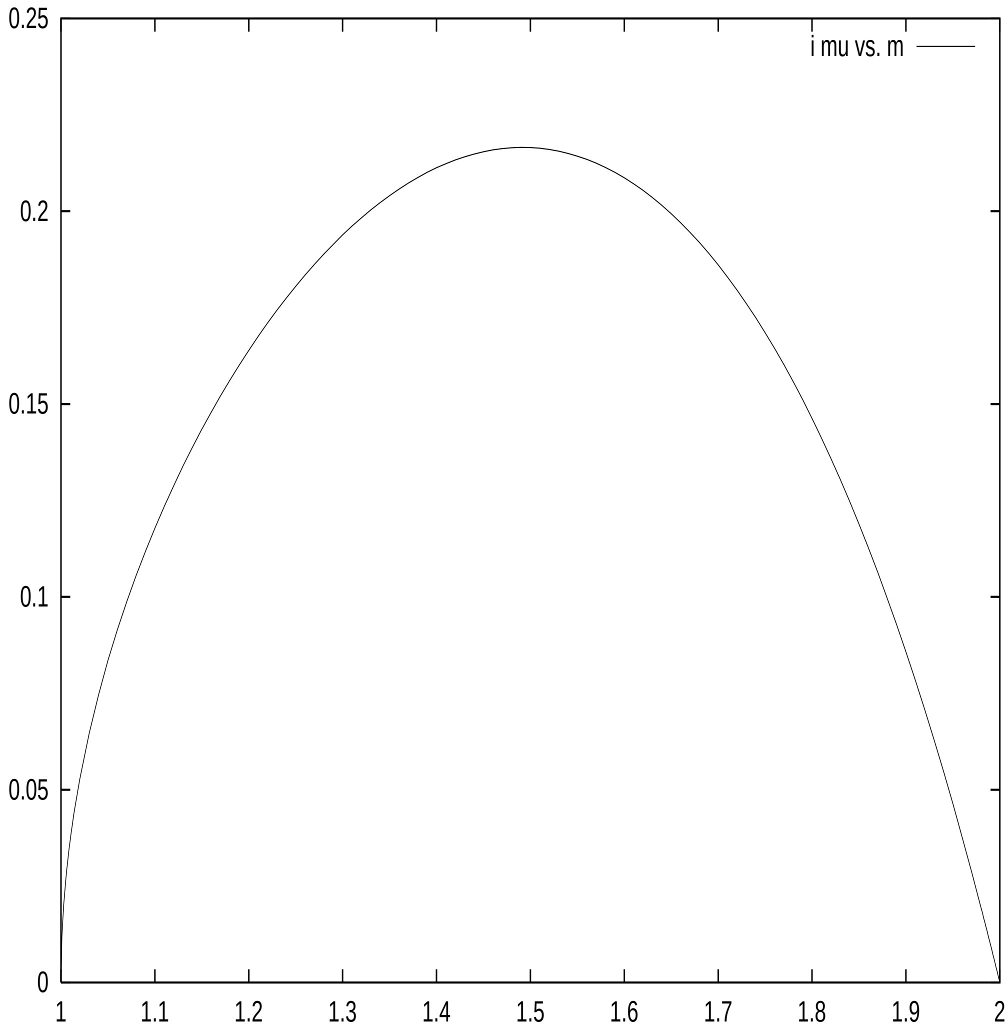

Figure 4: Imμ as a function of m for the odderon eigenvalue 1 in the interval $1 \leq m \leq 2$.[Here Re$\mu = 0$].

for the coefficients of the poles. Then, we impose the validity of the Baxter equation at infinity which gives further linear constraints on the pole coefficients.

The auxiliary functions (45) take here the form

$$
\begin{aligned}
f_3\,(x;\, m,\, \mu, q_4) &= \sum_{l=0}^{\infty} \left[\frac{a_l(m,\, \mu, q_4)}{(x-l)^3} + \frac{b_l(m,\, \mu, q_4)}{(x-l)^2} + \frac{c_l(m,\, \mu, q_4)}{x-l}\right] , \\
f_2\,(x;\, m,\, \mu, q_4) &= \sum_{l=0}^{\infty} \left[\frac{a_l(m,\, \mu, q_4)}{(x-l)^2} + \frac{b_l(m,\, \mu, q_4)}{x-l}\right] , \\
f_1\,(x;\, m,\, \mu, q_4) &= \sum_{l=0}^{\infty} \frac{a_l(m,\, \mu, q_4)}{x-l} \; .
\end{aligned}
\tag{111}
$$

Imposing the Baxter equations **at** the poles $x = l \;, l = 0, 1, 2, 3, \ldots$ yields the recurrence relations:

$$
\begin{aligned}
&(l+1)^4\, a_{l+1}(m,\, \mu, q_4) = \left[2\, l^4 + m(m-1)\, l^2 - \mu\, l \, + q_4\right]\, a_l(m,\, \mu, q_4) - (l-1)^4\, a_{l-1}(m,\, \mu, q_4) \; , \\
&(l+1)^4\, b_{l+1}(m,\, \mu, q_4) = \left[2\, l^4 + m(m-1)\, l^2 - \mu\, l + q_4\right]\, b_l(m,\, \mu, q_4) - (l-1)^4\, b_{l-1}(m,\, \mu, q_4) \\
&+ \left[8\, l^3 + 2m(m-1)\, l - \mu\right]\, a_l(m,\, \mu, q_4) - 4\, (l+1)^3\, a_{l+1}(m,\, \mu, q_4) - 4\, (l-1)^3\, a_{l-1}(m,\, \mu, q_4) \; , \\
&(l+1)^4\, c_{l+1}(m,\, \mu, q_4) = \left[2\, l^4 + m(m-1)\, l^2 - \mu\, l + q_4\right]\, c_l(m,\, \mu, q_4) - (l-1)^4\, c_{l-1}(m,\, \mu, q_4) \\
&+ \left[8\, l^3 + 2m(m-1)\, l - \mu\right]\, b_l(m,\, \mu, q_4) - 4\, (l+1)^3\, b_{l+1}(m,\, \mu, q_4) - 4\, (l-1)^3\, b_{l-1}(m,\, \mu, q_4) \\
&-6(l+1)^2\, a_{l+1}(m,\, \mu, q_4) - 6(l-1)^2\, a_{l-1}(m,\, \mu, q_4) + [12 l^2 + m(m-1)] a_l(m,\, \mu, q_4) \; .
\end{aligned}
\tag{112}
$$

We choose,

$$
a_0(m,\, \mu, q_4) = 1 \quad , \quad b_0 = 0 \quad , \quad c_0 = 0 \; , \tag{113}
$$

and **all** the coefficients $a_l(m,\, \mu, q_4)$, $b_l(m,\, \mu, q_4)$ and $c_l(m,\, \mu, q_4)$ become uniquely determined by eqs.(112). In particular,

$$
\begin{aligned}
&a_1(m,\, \mu, q_4) = q_4 \quad , \quad b_1(m,\, \mu, q_4) = -4q_4 - \mu \; , \\
&c_1(m,\, \mu, q_4) = 10\, q_4 + 4\mu + m(m-1) \; .
\end{aligned}
\tag{114}
$$

Taking linear combinations of $f_1\,(\pm x;\, m,\, \pm\mu, q_4)\,, f_2\,(x;\, m,\, \mu, q_4)$ and $f_3\,(x;\, m,\, \mu, q_4)$ as in eqs.(47) and (50) we form three independent solutions of the Baxter equation (110). We have,

$$
\begin{aligned}
Q^{(3)}\,(x;\, m,\, \mu, q_4) &\equiv f_3\,(x;\, m,\, \mu, q_4) + \alpha_1(m, \mu, q_4)\; f_2\,(x;\, m,\, \mu, q_4) + \alpha_2(m, \mu, q_4)\; f_1\,(x;\, m,\, \mu, q_4) \; , \\
Q^{(2)}\,(x;\, m,\, \mu, q_4) &\equiv f_2\,(x;\, m,\, \mu, q_4) + \gamma_1(m, \mu, q_4)\; f_1\,(x;\, m,\, \mu, q_4) + \\
&+ \gamma_2(m, \mu, q_4)\; f_1\,(-x;\, m,\, -\mu, q_4) \; .
\end{aligned}
\tag{115}
$$

The solution $Q^{(1)}\,(x;\, m,\, \mu, q_4)$ is proportional to $Q^{(2)}\,(-x;\, m,\, -\mu, q_4)$.

The recurrence relations for the coefficients of the poles guarantee that $B_4\,(x;\, m, \mu, q_4)$ is an entire function in general non-zero. More precisely, we find from eqs.(110) and (111) that $B_4\,(x;\, m, \mu, q_4) = k_1 + k_2\, x$, where k_1 and k_2 are some constants. These constants vanish and the Baxter equation is fulfilled **provided** the coefficients of x^{-1} and x^{-2} vanish for large x in the Q's of eq.(115). The coefficients of x^{-1} and x^{-2} in eqs.(123) being nonzero, the auxiliary functions $f_1(x;\, m,\, \mu, q_4)$, $f_2(x;\, m,\, \mu, q_4)$ and $f_3(x;\, m,\, \mu, q_4)$ **are not** solutions of the Baxter equation.

The coefficients $\alpha_1(m, \mu, q_4),\ \alpha_2(m, \mu, q_4),\ \gamma_1(m, \mu, q_4)$ and $\gamma_2(m, \mu, q_4)$ are chosen imposing the Baxter equation at infinity. We present the linear equations on $\alpha_1(m, \mu, q_4),\ \alpha_2(m, \mu, q_4)$, $\gamma_1(m, \mu, q_4)$ and $\gamma_2(m, \mu, q_4)$ and their explicit solutions in the appendix A.

There is one independent linear relation (54) between the Baxter solutions $Q^{(t)}$ for $n = 4$

$$\left[\delta^{(2)}(\mu,\, m, q_4) + \pi \cot \pi x\right] Q^{(2)}(x;\, m,\, \mu, q_4) = Q^{(3)}(x;\, m,\, \mu, q_4) + \alpha^{(2)}(\mu,\, m, q_4) Q^{(2)}(-x;\, m,\, -\mu, q_4) \ .$$

where $\delta^{(2)}(\mu,\, m, q_4) = \alpha_1(\mu,\, m, q_4) - \gamma_1(\mu,\, m, q_4)$. An explicit proof of this equation is given in the Appendix B.

The energy of the four Reggeons state is given by[19]

$$E = \lim_{x, x^* \to i} \left\{ \frac{\partial}{\partial x} \ln\left[(x-1)^3\, x^4\, Q(x;\, m,\, \mu, q_4)\right] + \frac{\partial}{\partial x^*} \ln\left[(x^*-1)^3\, x^{*4}\, Q(x^*;\, \widetilde{m},\, -\mu, q_4)\right] \right\} . \tag{116}$$

Again, the energy is expressed in terms of the behavior of the Baxter function near $x = 1$. That is,

$$\begin{aligned} Q^{(3)}(x;\, m,\, \mu, q_4) &\overset{x \to 1}{=} \frac{q_4}{(x-1)^3}\left[1 - (x-1)\left(4 + \frac{\mu}{q_4} + \alpha_1(m, \mu, q_4)\right) + \mathcal{O}\left([x-1]^2\right)\right] \\ \pi \cot[\pi x]\, Q^{(2)}(x;\, m,\, \mu, q_4) &\overset{x \to 1}{=} -\frac{q_4}{(x-1)^3}\left[1 - (x-1)\left(4 + \frac{\mu}{q_4} + \gamma_1(m, \mu, q_4)\right) + \mathcal{O}\left([x-1]^2\right)\right] \end{aligned} \tag{117}$$

We obtain for the energy of the three solutions from eqs.(116) and (117),

$$\begin{aligned} E^{(3)}(m, \widetilde{m}, \mu, q_4) &= -\alpha_1(m, \mu, q_4) - \alpha_1(\widetilde{m}, -\mu, q_4) , \\ E^{(2)}(m, \widetilde{m}, \mu, q_4) &= -\gamma_1(m, \mu, q_4) - \gamma_1(\widetilde{m}, -\mu, q_4) . \end{aligned} \tag{118}$$

The eigenvalue condition

$$\alpha_1(\mu,\, m, q_4) - \gamma_1(\mu,\, m, q_4) = 0 \quad , \quad \alpha_1(-\mu,\, \widetilde{m}, q_4) - \gamma_1(-\mu,\, \widetilde{m}, q_4) = 0 \tag{119}$$

guarantees that the energy is the same for the two independent Baxter solutions,

$$E^{(3)}(m, \widetilde{m}, \mu, q_4) = E^{(2)}(m, \widetilde{m}, \mu, q_4)$$

Eqs.(119) fix the possible values of μ and q_4 for given $m, \widetilde{m}$.

We find from the above equations for $m = \widetilde{m} = 1/2$ the first roots numerically as,

$$\mu = 0 \quad , \quad q_4 = 0.1535892\ldots \quad , \quad E = -1.34832\ldots \quad ,$$

$$\mu = 0.73833\ldots \quad , \quad q_4 = -0.3703\ldots \quad , \quad E = 2.34105\ldots \quad ,$$

$$\mu = 0\ldots \quad , \quad q_4 = -0.292782\ldots \quad , \quad E = 2.756624\ldots \quad ,$$

$$\mu = 1.4100\ldots \quad , \quad q_4 = 0.73852\ldots \quad , \quad E = 3.3581\ldots \quad ,$$

$$\mu = 0\ldots \quad , \quad q_4 = 1.79992\ldots \quad , \quad E = 5.67117\ldots \quad . \tag{120}$$

We plot in fig.7 the eigenvalue equations (119) in the μ, q_4-plane for $m = \widetilde{m} = \frac{1}{2}$. The curves intersect at the eigenstates (120).

We find for the first eigenvalues for $m = 0$, $\widetilde{m} = 1$ corresponding to $n = -1$ in eq.(77).[$n = 1$ gives the same state with $m \leftrightarrow \widetilde{m}$].

$$\mu = 0 \quad , \quad q_4 = 0.12167\ldots \quad , \quad E = -2.0799\ldots \quad ,$$

$$\mu = 0.51214\ldots \quad , \quad q_4 = -0.33288\ldots \quad , \quad E = 2.2007\ldots \quad ,$$

$$\mu = 0 \quad , \quad q_4 = -0.2905426\ldots \quad , \quad E = 2.441210\ldots \quad . \tag{121}$$

Therefore, the **ground state** of the quarteton corresponds to $m = 0$, $\widetilde{m} = 1$. Its energy, $E = -2.0799\ldots$ is **below** the energy $E = -1.34832\ldots$ of the lowest state with $m = \widetilde{m} = \frac{1}{2}$.

We plot in fig. 8 the eigenvalue equations (119) in the μ, q_4-plane for $m = 0$, $\widetilde{m} = 1$. The curves intersect at the eigenstates (121). It should be noticed that the curves 7 and 8 turn to be qualitatively similar. The eigenstates with $m = 0$, $\widetilde{m} = 1$ follow from those with $m = \widetilde{m} = \frac{1}{2}$ by analytic continuation in m, $\widetilde{m}$.

We have followed the first eigenvalue as a function of m for $0 < m < \frac{1}{2}$. The result is plotted in fig. 6. Contrary to the odderon case, we find that the energy eigenvalue does not vanish for $m = 0$. The energy decreases with m for $0 < m < \frac{1}{2}$ and takes the value $E = -2.0799\ldots$ at $m = 0$ while q_4 takes there the value $0.12167\ldots$. μ vanishes for this eigenvalue for all m [see fig. 5].

Near $m = 2$ the lowest energy state of four reggeons near m=2. The energy goes to minus infinity and q_4 vanishes for $m \to 2$ for the lowest eigenstate of four reggeons. [q_3 is zero for all m in this state]. We find the following behaviour,

$$E_1 \overset{m \to 2}{=} \frac{4}{m-2} + 2 + 2 - m + \mathcal{O}\left[(m-2)^2\right] \quad , \quad q_4 \overset{m \to 2}{=} \frac{1}{4}(m-2)^2 + \mathcal{O}\left[(m-2)^3\right] \; . \tag{122}$$

Nota Added: After completion of this work we have seen the preprint hep-th/0204124 by S. É. Derkachov et al. studying similar problems. It is stated there that the ground state for three and four reggeons corresponds to zero conformal spin $m - \widetilde{m} = 0$. We show here that the lowest energy eigenvalue corresponds in both cases to conformal spin $|m - \widetilde{m}| = 1$.

A Asymptotic constraints on the solutions of the Baxter equation. The quarteton case.

We impose here the Baxter equation at infinity on the solutions for four reggeons. As derived in sec. VI, the coefficients of x^{-1} and x^{-2} in $Q^{(3)}(x;\, m,\, \mu, q_4)$ and $Q^{(2)}(x;\, m,\, \mu, q_4)$ for $x \to \infty$ must vanish.

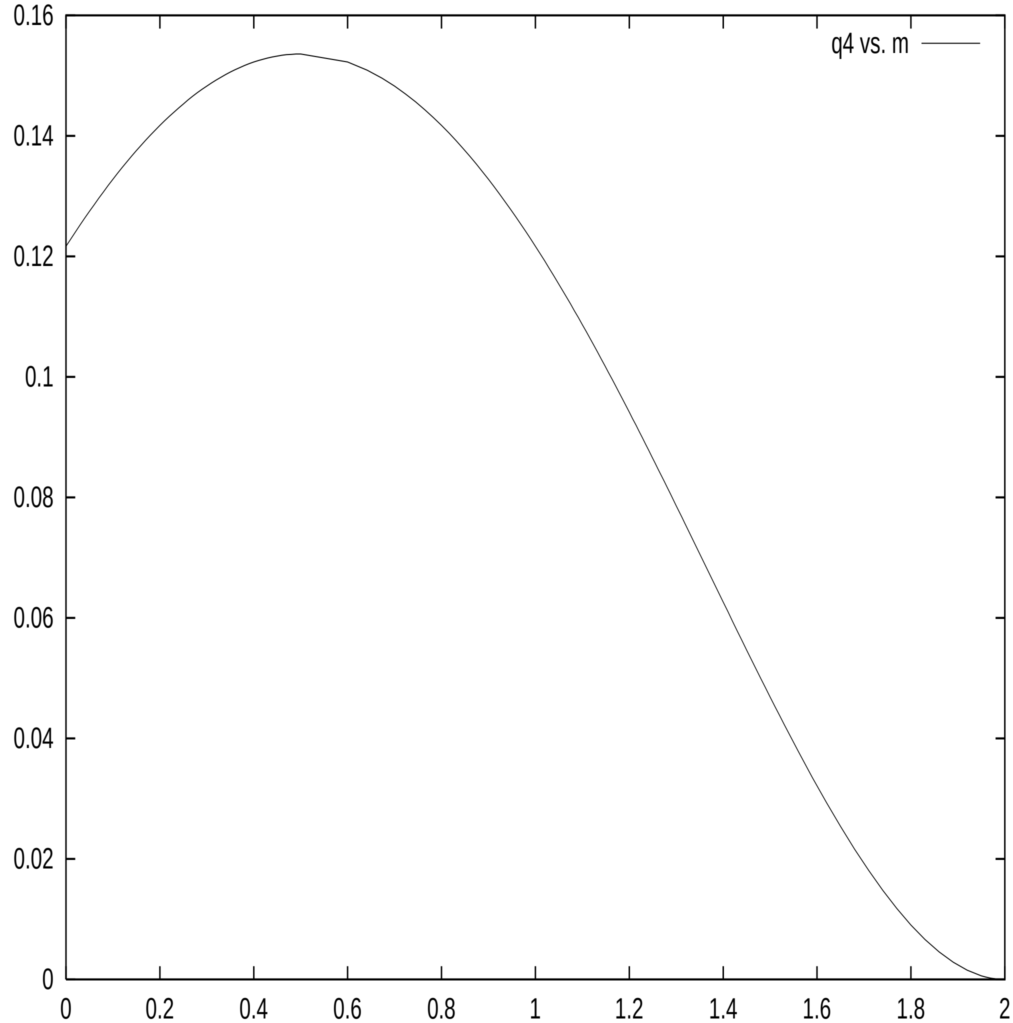

Figure 5: The q_4 as a function of m for the quarteton eigenvalue 1 in the interval $0 < m < 2$. We have for this eigenvalue $\mu = 0$ for all m.

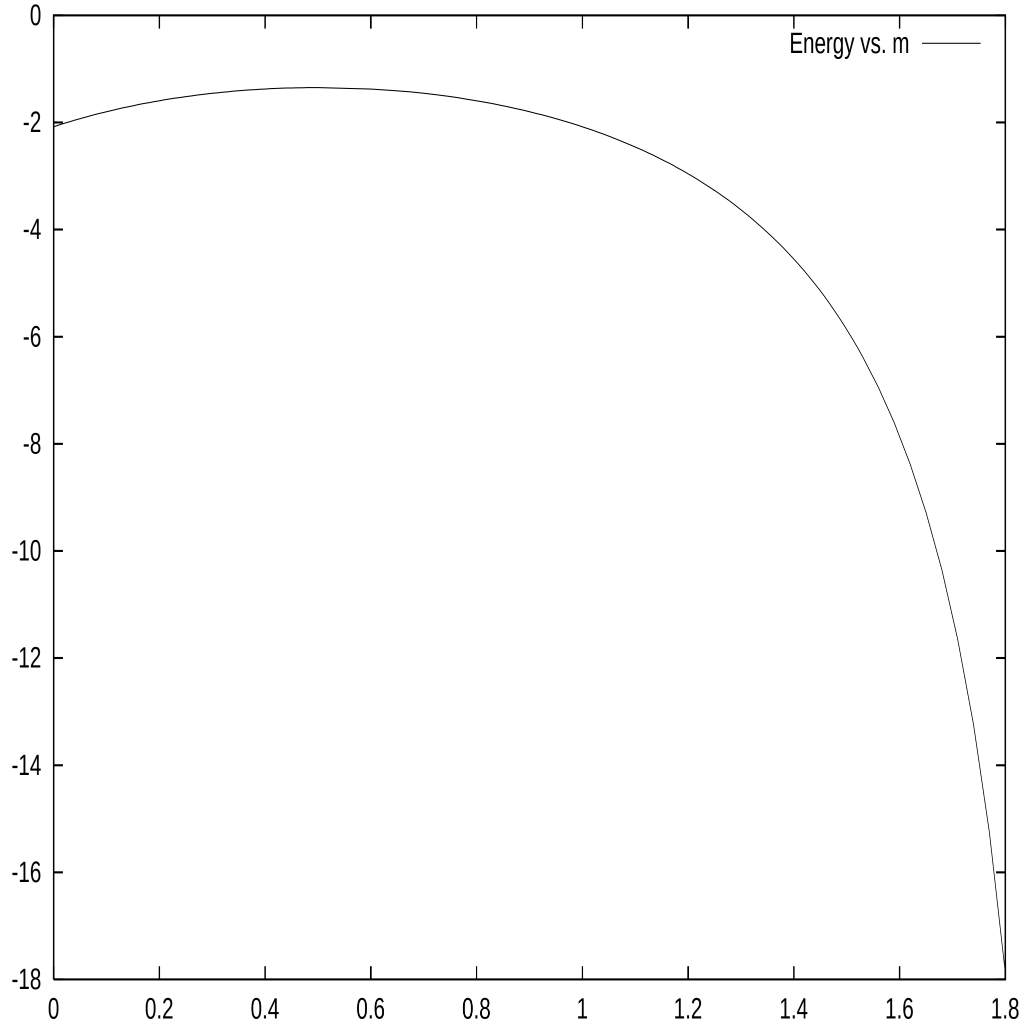

Figure 6: The energy as a function of m for the quarteton eigenvalue 1 in the interval $0 < m < 2$. We have for this eigenvalue $\mu = 0$ for all m.

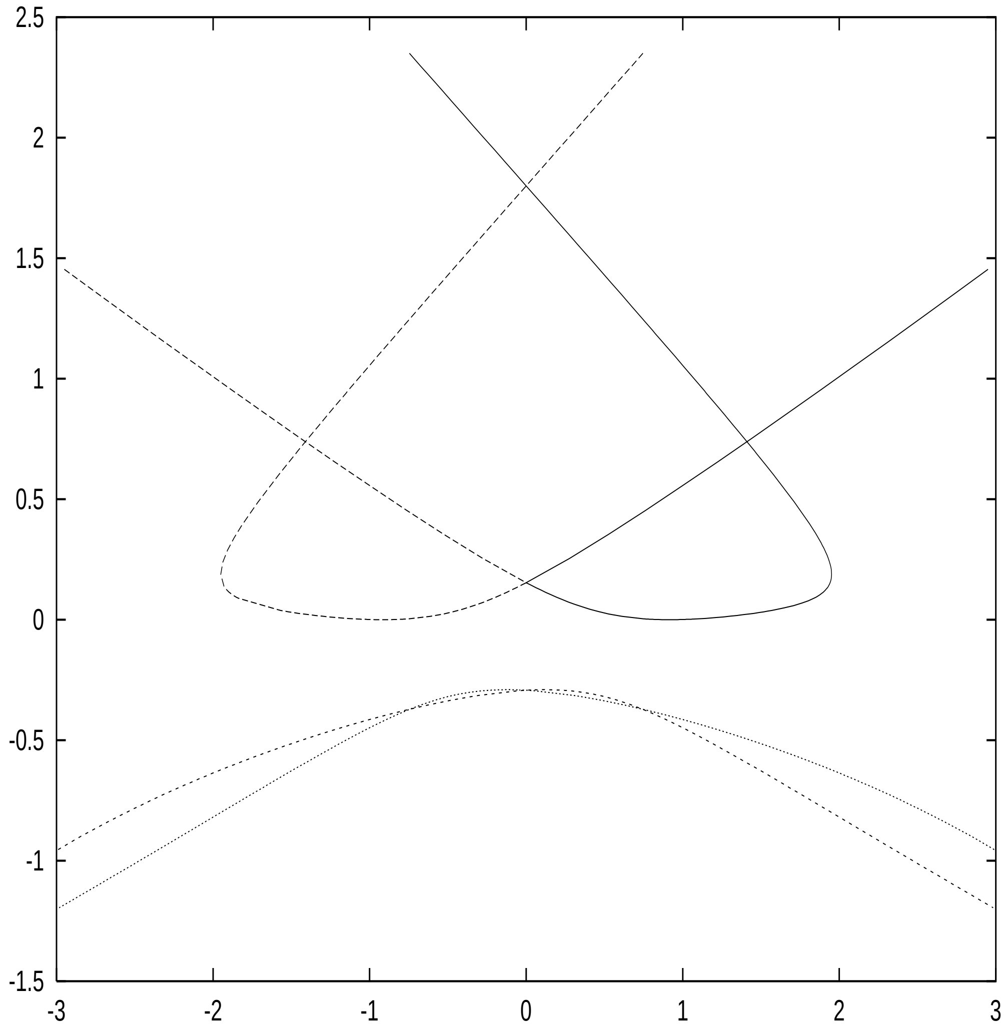

Figure 7: The quarteton eigenvalue equations $\alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4) = 0$ and $\alpha_1(-\mu, \widetilde{m}, q_4) - \gamma_1(-\mu, \widetilde{m}, q_4) = 0$ for $m = \widetilde{m} = \frac{1}{2}$ in the μ, q_4-plane

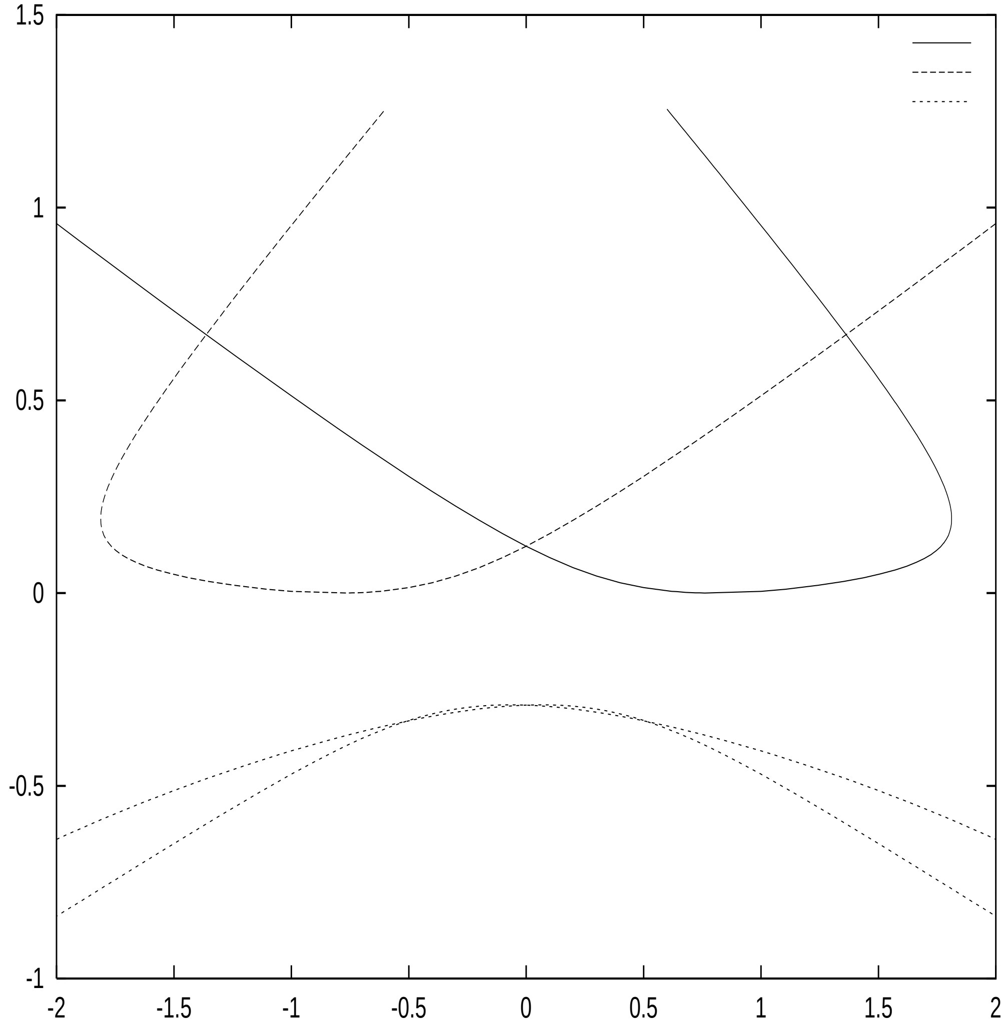

Figure 8: The quarteton eigenvalue equations $\alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4) = 0$ and $\alpha_1(-\mu, \widetilde{m}, q_4) - \gamma_1(-\mu, \widetilde{m}, q_4) = 0$ for $m = 0,\ \widetilde{m} = 1$ in the $\mu,\ q_4$-plane

For large x the series (111) yield the following formal expansions,

$$f_3\,(x;\,m,\,\mu,q_4) \stackrel{|x|\to\infty}{=} \frac{1}{ix}\sum_{r=0}^{\infty} c_r(m,\,\mu,q_4) + \frac{1}{(ix)^2}\sum_{r=0}^{\infty}\left[b_r(m,\,\mu,q_4) - r\,c_r(m,\,\mu,q_4)\right] + \mathcal{O}\left(\frac{1}{x^3}\right)\,,$$
$$f_2\,(x;\,m,\,\mu,q_4) \stackrel{|x|\to\infty}{=} \frac{1}{ix}\sum_{r=0}^{\infty} b_r(m,\,\mu,q_4) + \frac{1}{(ix)^2}\sum_{r=0}^{\infty}\left[a_r(m,\,\mu,q_4) - r\,b_r(m,\,\mu,q_4)\right] + \mathcal{O}\left(\frac{1}{x^3}\right)\,,$$
$$f_1\,(x;\,m,\,\mu,q_4) \stackrel{|x|\to\infty}{=} \frac{1}{ix}\sum_{r=0}^{\infty} a_r(m,\,\mu,q_4) - \frac{1}{(ix)^2}\sum_{r=0}^{\infty} r\,a_r(m,\,\mu,q_4) + \mathcal{O}\left(\frac{1}{x^3}\right)\,. \tag{123}$$

For $Q^{(3)}\,(x;\,m,\,\mu,q_4)$ we find the conditions,

$$\alpha_1(m,\mu,q_4)\,\sum_{r=0}^{\infty} b_r(m,\,\mu,q_4) + \alpha_2(m,\mu,q_4)\,\sum_{r=0}^{\infty} a_r(m,\,\mu,q_4) = -\sum_{r=0}^{\infty} c_r(m,\,\mu,q_4)\,,$$
$$\alpha_1(m,\mu,q_4)\,\sum_{r=0}^{\infty}\left[a_r(m,\,\mu,q_4) - r\,b_r(m,\,\mu,q_4)\right] - \alpha_2(m,\mu,q_4)\,\sum_{r=0}^{\infty} r\,a_r(m,\,\mu,q_4) =$$
$$= \sum_{r=0}^{\infty}\left[r\,c_r(m,\,\mu,q_4) - b_r(m,\,\mu,q_4)\right]\,. \tag{124}$$

Imposing the same constraint on $Q^{(2)}\,(x;\,m,\,\mu,q_4)$ yields,

$$\gamma_1(m,\mu,q_4)\,\sum_{r=0}^{\infty} a_r(m,\,\mu,q_4) - \gamma_2(m,\mu,q_4)\,\sum_{r=0}^{\infty} a_r(m,-\mu,q_4) = -\sum_{r=0}^{\infty} b_r(m,\,\mu,q_4)\,,$$
$$\gamma_1(m,\mu,q_4)\,\sum_{r=0}^{\infty} r\,a_r(m,\,\mu,q_4) + \gamma_2(m,\mu,q_4)\,\sum_{r=0}^{\infty} r\,a_r(m,-\mu,q_4) =$$
$$= \sum_{r=0}^{\infty}\left[a_r(m,\,\mu,q_4) - r\,b_r(m,\,\mu,q_4)\right] \tag{125}$$

Eqs.(124) can be easily solved yielding,

$$\alpha_1(m,\mu,q_4) = \frac{1}{\Delta_\alpha(m,\mu,q_4)}\left\{\sum_{r=0}^{\infty} r\,a_r(m,\,\mu,q_4)\sum_{n=0}^{\infty} c_n(m,\,\mu,q_4)\right.$$
$$\left.-\sum_{n=0}^{\infty} a_n(m,\,\mu,q_4)\sum_{r=0}^{\infty}\left[r\,c_r(m,\,\mu,q_4) - b_r(m,\,\mu,q_4)\right]\right\}\,,$$
$$\alpha_2(m,\mu,q_4) = \frac{1}{\Delta_\alpha(m,\mu,q_4)}\left\{\sum_{n=0}^{\infty} a_n(m,\,\mu,q_4)\sum_{r=0}^{\infty}\left[a_r(m,\,\mu,q_4) - r\,b_r(m,\,\mu,q_4)\right]\right.$$
$$\left.-\sum_{r=0}^{\infty} b_r(m,\,\mu,q_4)\sum_{n=0}^{\infty} n\,a_n(m,\,\mu,q_4)\right\} \tag{126}$$

where

$$\Delta_\alpha(m,\mu,q_4) \equiv \sum_{n=0}^{\infty} a_n(m,\,\mu,q_4)\sum_{r=0}^{\infty}\left[r\,b_r(m,\,\mu,q_4) - a_r(m,\,\mu,q_4)\right] - \sum_{r=0}^{\infty} b_r(m,\,\mu,q_4)\sum_{n=0}^{\infty} n\,a_n(m,\,\mu,q_4)\,.$$

We analogously obtain from eqs. (125)

$$\gamma_1(m,\mu,q_4) = \frac{1}{\Delta_\gamma(m,\mu,q_4)}\left\{\sum_{n=0}^{\infty} a_n(m,-\mu,q_4)\sum_{r=0}^{\infty}\left[a_r(m,\mu,q_4) - r\,b_r(m,\mu,q_4)\right]\right.$$

$$- \sum_{n=0}^{\infty} b_n(m,\mu,q_4) \sum_{r=0}^{\infty} r\, a_r(m,-\mu,q_4) \Bigg\} ,$$

$$\gamma_2(m,\mu,q_4) = \frac{1}{\Delta_\gamma(m,\mu,q_4)} \left\{ \sum_{n=0}^{\infty} a_n(m,\mu,q_4) \sum_{r=0}^{\infty} [a_r(m,\ \mu,q_4) - r\, b_r(m,\ \mu,q_4)] \right.$$

$$\left. + \sum_{n=0}^{\infty} b_n(m,\mu,q_4) \sum_{r=0}^{\infty} r\, a_r(m,\mu,q_4) \right\} , \tag{127}$$

where,

$$\Delta_\gamma(m,\mu,q_4) \equiv \sum_{n=0}^{\infty} a_n(m,\mu,q_4) \sum_{r=0}^{\infty} r\, a_r(m,-\mu,q_4) + \sum_{n=0}^{\infty} a_n(m,-\mu,q_4) \sum_{r=0}^{\infty} r\, a_r(m,\mu,q_4) .$$

B Linear Relations between the solutions of the Baxter equation for the quarteton case.

We present here an explicit proof of the linear recurrence relations (54) for the quarteton case. The Baxter solutions $Q^{(3)}(x;\, m,\, \mu, q_4)$ and $Q^{(2)}(x;\, m,\, \mu, q_4)$ are related by

$$[\alpha_1(\mu,\, m, q_4) - \gamma_1(\mu,\, m, q_4) + \pi \cot \pi x]\, Q^{(2)}(x;\, m,\, \mu, q_4)$$

$$= Q^{(3)}(x;\, m,\, \mu, q_4) - \gamma_2(\mu,\, m, q_4) Q^{(2)}(-x;\, m,\, -\mu, q_4) \ . \tag{128}$$

This is eq.(54) for $n = 4,\ r = 2$ and $\delta^{(2)}(\mu,\, m, q_4) = \alpha_1(\mu,\, m, q_4) - \gamma_1(\mu,\, m, q_4),\ \alpha^{(2)}(\mu,\, m, q_4) = -\gamma_2(\mu,\, m, q_4)$. Notice that $Q^{(1)}(x;\, m,\, \mu, q_4)$ is proportional to $Q^{(2)}(-x;\, m,\, -\mu, q_4)$ according to eq.(53) and therefore eq.(128) is the only independent three-terms linear relation between Baxter solutions.

In order to prove this relations we compute its triple, double and simple poles at $x = l \in \mathcal{Z}$. In the course of these calculations we use that $\pi \cot \pi x$ as a function of x has unit residue at all these integer poles.

Moreover, we have from eqs.(111) and (115) that

$$Q^{(2)}(x;\, m,\, \mu, q_4) \overset{x \to r \geq 0}{=} \frac{a_r(m,\, \mu, q_4)}{(x-r)^2} + \frac{b_r(m,\, \mu, q_4) + \gamma_1(\mu, ;\, m, q_4)\, a_r(m,\, \mu, q_4)}{x - r}$$

$$+ A_r(m,\, \mu, q_4) + \mathcal{O}(x - r)$$

$$Q^{(2)}(x;\, m,\, \mu, q_4) \overset{x \to -r < 0}{=} -\gamma_2(\mu,\, m, q_4)\, \frac{a_r(m,\, -\mu, q_4)}{x + r} + \hat{A}_r(m,\, \mu, q_4) + \mathcal{O}(x + r) \tag{129}$$

Inserting eqs.(129) into the Baxter equation (110) yields the recurrence relations

$$(r+1)^4 A_{r+1}(m,\, \mu, q_4) = \left[2\, r^4 + m(m-1)\, r^2 - \mu\, r + q_4\right] A_r(m,\, \mu, q_4) - (r-1)^4 A_{r-1}(m,\, \mu, q_4)$$

$$+ \left[8\, r^3 + 2m(m-1)\, r - \mu\right] b_r(m,\, \mu, q_4) - 4\,(r+1)^3\, b_{r+1}(m,\, \mu, q_4) - 4\,(r-1)^3\, b_{r-1}(m,\, \mu, q_4)$$

$$-6(r+1)^2\, a_{r+1}(m,\ \mu, q_4) - 6(r-1)^2\, a_{r-1}(m,\ \mu, q_4) + [12r^2 + m(m-1)]a_r(m,\ \mu, q_4)$$

$$+\gamma_1(\mu, ;\ m, q_4) \left\{ \left[8\, r^3 + 2m(m-1)\, r - \mu \right]\ a_r(m,\ \mu, q_4) \right.$$

$$\left. -4\,(r+1)^3\, a_{r+1}(m,\ \mu, q_4) - 4\,(r-1)^3\, a_{r-1}(m,\ \mu, q_4) \right\}\ . \tag{130}$$

and

$$(r+1)^4\, \hat{A}_{r+1}(m,\ \mu, q_4) = \left[2\, r^4 + m(m-1)\, r^2 + \mu\, r + q_4 \right]\ \hat{A}_r(m,\ \mu, q_4) - (r-1)^4\, \hat{A}_{r-1}(m,\ \mu, q_4)$$

$$+\gamma_2(\mu, ;\ m, q_4) \left\{ \left[8\, r^3 + 2m(m-1)\, r + \mu \right]\ a_r(m,\ -\mu, q_4) \right.$$

$$\left. -4\,(r+1)^3\, a_{r+1}(m,\ -\mu, q_4) - 4\,(r-1)^3\, a_{r-1}(m,\ -\mu, q_4) \right\}\ . \tag{131}$$

These recurrence relations can be solved in terms of the coefficients $a_r,\ b_r$ and c_r using eq.(112). One has to take into account that $A_r(m,\ \mu, q_4)$ and $\hat{A}_r(m,\ \mu, q_4)$ do not obey the initial conditions (113). We find,

$$A_r(m,\ \mu, q_4) = c_r(m,\ \mu, q_4) + \gamma_1(\mu, ;\ m, q_4)\ b_r(m,\ \mu, q_4) + A_0(m,\ \mu, q_4)\ a_r(m,\ \mu, q_4)$$

$$\hat{A}_r(m,\ \mu, q_4) = \gamma_2(\mu, ;\ m, q_4)\ b_r(m,\ -\mu, q_4) + A_0(m,\ \mu, q_4)\ a_r(m,\ -\mu, q_4) \tag{132}$$

Using eqs.(111), (115) and (132) we find that eq.(128) holds at all of its poles. Therefore, the r. h. s. and the l. h. s. of eq.(128) can only differ on a entire function. Taking into account the asymptotic behaviour of the Baxter functions we conclude that this entire function is identically zero.

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